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On Strong Uniform Distribution, III

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Abstract. We construct infinite dimensional chains that are $\mathscr{L}^1$ good for almost sure convergence, which settles a question raised in this journal [7] and earlier in [6] by R. Nair. In [7] it was stated that the construction proposed in [4] was invalid. We complete the construction proposed in [4], where it is true that a piece of proof was forgotten. The technic remains the same and the completion of the proof rather natural.

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1. Introduction

A chain $\mathscr{C}$ is a multiplicative sub-semigroup of the one of positive integers $\mathbb{N}$. We say a sequence $\mathbf{p} = (p_k)$ of primes generates the chain $\mathscr{C}$ if $\mathscr{C} = \{\prod_{k=1}^{J} p_k^{\alpha_k} : \alpha_k \geqslant 0, J \geqslant 1\}$. A chain is of finite dimension (abbreviated "an FD chain") if the sequence of primes generating it is finite; else, it is infinite dimensional (abbreviated "ID chain").

Throughout, $(\mathbb{T}, \lambda)$ denotes the reals mod 1 with Lebesgue measure. We shall say that a chain $\mathscr{C}$ is good if, once one orders $\mathscr{C} = \{a_1 < a_2 < \cdots\}$, it holds that for any $f \in \mathscr{L}^1(\mathbb{T}, \lambda)$,

$$\frac{1}{k}\sum_{j=1}^{k} f(a_j x \bmod 1) \to \int_0^1 f d\lambda \quad \text{for } \lambda - \text{a.e. } x \in \mathbb{T}. \tag{1}$$

Else we say $\mathscr{C}$ is a bad chain.

Nair [6, 7] asks twice for the existence of a good ID chain. He proves in [6] that an FD chain is always good, using the multidimensional ergodic theorem ([1], [3]) for $\mathbb{N}^d$-actions. It is also known ([2], [5]) that taking all the primes generates a bad ID chain, with counter-examples to almost sure convergence in (1) for some $f \in \mathscr{L}^\infty(\mathbb{T}, \lambda)$.

In [7] one can find on page 342, a few lines after formula [7, (1.3)], the following (we have adapted the reference numbering):

"*In* [6] *the author raised the question whether the condition in his theorem that the set* $p_1, \ldots, p_d$ *be finite is necessary. This question remains open despite the invalid construction of a putative such infinite set in* [4]."

A PhD student of the author, Vincent Chaumoître, has, following [7], made a precise rereading of [4], and pointed out to the author the precise spot where [4] was uncomplete. In fact, the displayed formula between [4, (5)] and [4, ($P6$)] is only correct in the FD case, and hence the reduction of [4, ($P2$)] to [4, ($P6$)] is not valid in the ID case. The place where this omission occurs in [4] corresponds to the part of the paper devoted to show how the result from [6] could be recovered using Tempelman's ergodic theorem, giving a simple proof that an FD chain is good [4, Corollary 1] (cf. [6] for a different approach).

The present note completes the gap, using exactly the same ideas as in [4] to produce a complete proof of the following:

Theorem 1. *There exist good ID chains.*

We will present the completed argumentation omitting the ergodic theoretic preliminaries for which we refer to [4]. Let us mention by the way that the construction of bad ID chains in [4] is perfectly valid.

2. Good ID Chains!

2.1. Semigroup actions and Tempelman's conditions. We shall make essential use of the following abelian semi-group endowed with its counting measure (for a subset T, $\#T$ denotes its cardinality):

$$l_0(\mathbb{N}) := \{(\alpha_i)_{i\geqslant 1} : \alpha_i \in \mathbb{N},\ \exists j,\quad i>j \Rightarrow \alpha_i = 0\}.$$

Given an integer q, we identify $\mathbb{N}^q$ with a sub-semigroup of $\mathbb{N}^{q+1}$ and the later with one of $l_0(\mathbb{N})$ via the following embeddings;

$$\begin{array}{ccccc} \mathbb{N}^q & \hookrightarrow & \mathbb{N}^{q+1} & \hookrightarrow & l_0(\mathbb{N}) \\ (\alpha_1,\ldots,\alpha_q) & \mapsto & (\alpha_1,\ldots,\alpha_q,0) & \mapsto & (\alpha_1,\ldots,\alpha_q,0,0,\ldots). \end{array}$$

For an integer p, we define $T_p : X \to X$ by $T_p x = px \bmod 1$. It is standard that the system $(\mathbb{T},\lambda,T_p)$ is metrically conjugated to a one sided Bernoulli shift which is ergodic [3]. It is standard also that $T_p \circ T_q = T_q \circ T_p = T_{pq}$, whence given a sequence of integers $(p_k)_{k\geqslant 1}$, we define an action Γ of $l_0(\mathbb{N})$ on $(\mathbb{T},\lambda,T_p)$ by

$$\Gamma((\alpha_k)) := \bigcirc_{k\geqslant 1} T_{p_k}^{\alpha_k},$$

where $T_{p_k}^0$ is the identity map. Given any sequence $(T(n))$ of subsets of $l_0(\mathbb{N})$, we will consider the following multiple condition (P);

$$\begin{cases} (P1):\ 0<\#T(n)<\infty, \\ (P2):\ \forall\gamma\in l_0(\mathbb{N}), \lim_n \#((T(n)+\gamma)\Delta T(n))/\#T(n) = 0, \\ (P3):\ T(n)\subset T(n+1), n\geqslant 1, \\ (P4):\ \exists K_1<\infty, \forall N, \lim_n \#(T(N)+T(n))/\#T(n) \leqslant K_1, \\ (P5):\ \exists K_2<\infty, \forall n, \#(T(n)-T(n))/\#T(n) \leqslant K_2, \end{cases} \tag{P}$$

where $T(n)-T(n) := \{\alpha\in l_0(\mathbb{N}) : \exists\gamma\in T(n),\ \alpha+\gamma\in T(n)\}$.

Indeed, if $(T(n))$ satisfies (P), by Tempelman's Ergodic Theorem [3, p. 224], for any $f \in \mathscr{L}^1(\mu)$, the averages

$$\frac{1}{\#T(n)} \sum_{\alpha \in T(n)} f \circ \Gamma(\alpha)(x) \tag{2}$$

converge μ-a.e.. Moreover, the limit in (2) is, following the argument in [3, p. 206], or [8, Theorem 6.3.1], a Γ-invariant function, whence T_p-invariant for some $p \geqslant 2$, whence, by ergodicity of T_p, it is constant and must coincide with the expectation of f, as is standard.

When (2) holds we say that $(T(n))$ *is* $\mathscr{L}^1$ *good universal* (for $l_0(\mathbb{N})$ actions). We shall see in the next section that for some choice of $(T(n))$ averages in (1) and (2) coincide, therefore the condition (P) will be used to produce a good ID chain.

The same remarks can be stated for $\mathbb{N}^q$-actions.

2.2. Condition (P) for a pairwise coprime generated chain and the FD case. Let $p_1 < p_2 < \cdots$ be pairwise coprime integers generating the chain $\mathscr{C} = \{a_1 < a_2 < \cdots\}$. For given $q \geqslant 1$ and $n \in [1, \infty[$, we let

$$\begin{cases} T_q(n) := \{(\alpha_1, \ldots, \alpha_q) \in \mathbb{N}^q : \sum_{i=1}^q \alpha_i \log p_i \leqslant \log n\}, \\ T(n) := \{\alpha = (\alpha_i) \in l_0(\mathbb{N}) : \sum_{i \geqslant 1} \alpha_i \log p_i \leqslant \log n\}. \end{cases} \tag{3}$$

We notice that sequences (1) and (2) coincide for this choice of $(T(n))$ ($(T_q(n))$ in the FD case). For given $q \geqslant 1$, both $(T_q(n))$ and $(T(n))$ satisfy $(P1)$, $(P3)$, and $(P5)$ with $K_2 = 1$, because $T(n) - T(n) \subset T(n)$.

Moreover, since $T(n) \subset T(N) + T(n)$ (resp. $T_q(n) \subset T_q(N) + T_q(n)$), we have

$$\#(T(N) + T(n)) \leqslant \#T(n) + \textstyle\sum_{\gamma \in T(N)} \#((\gamma + T(n)) \backslash T(n))$$
$$(\text{resp. } \#(T_q(N) + T_q(n)) \leqslant \#T_q(n) + \textstyle\sum_{\gamma \in T_q(N)} \#((\gamma + T_q(n)) \backslash T_q(n))),$$

so we see that $(P2)$ implies $(P4)$ with $K_1 = 1$. Hence we deduce

Lemma 1. *The sequence* $(T(n))$ (*resp.* $(T_q(n))$) *defined by* (3) *is* $\mathscr{L}^1$ *good universal for* $l_0(\mathbb{N})$ *(resp.* $\mathbb{N}^q$*) actions whenever it satisfies* $(P2)$.

Given $\gamma = (\gamma_i) \in l_0(\mathbb{N})$ (resp. $\gamma \in \mathbb{N}^q$), we have

$$\#((T(n)+\gamma)\Delta T(n)) = \#(T(n) \backslash (T(n)+\gamma)) + \#((T(n)+\gamma) \backslash T(n))$$
$$(\textit{resp.}\, \#((T_q(n)+\gamma)\Delta T_q(n)) = \#(T_q(n) \backslash (T_q(n)+\gamma)) + \#((T_q(n)+\gamma) \backslash T_q(n))). \tag{4}$$

An elementary computation [5] shows that

$$\#T_q(n) \sim \frac{(\log n)^q}{q! \prod_{i=1}^q \log p_i}, \tag{5}$$

where $\sim$ means that the ratio of its left and right hand sides goes to 1 as n goes to ∞.

For any $\gamma \in \mathbb{N}^q, T_q(n) \backslash (T_q(n) + \gamma) = \{\alpha \in T_q(n) : \exists i : \alpha_i < \gamma_i\}$, so if we set

$$B_q(n,i) = \{\alpha \in T_q(n) : \alpha_i < \gamma_i\} \quad \text{and}$$
$$T_q^{(i)}(n) = \{(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_q) \in \mathbb{N}^{q-1} : \textstyle\sum_{\substack{j=1, \\ j \neq i}}^q \alpha_j \log p_j \leqslant \log n\},$$

then one observes that

$$\#(T_q(n)\backslash(T_q(n)+\gamma)) \leqslant \sum_{i=1}^{q} \#B_q(n,i) \leqslant \sum_{i=1}^{q} \gamma_i \#T_q^{(i)}(n),$$

whence by (5) we get at once that $\lim_n \#(T_q(n)\backslash T_q(n)+\gamma))\big/\#T_q(n)=0$. Secondly, we have that

$$\begin{aligned}(T_q(n)+\gamma)\backslash T_q(n) &= \{\alpha+\gamma : \textstyle\sum_i \alpha_i \log p_i \leqslant \log n \text{ and} \sum_i(\alpha_i+\gamma_i)\log p_i > \log n\} \\ &\subseteq \{\beta \in \mathbb{N}^q : \log n < \textstyle\sum_i \beta_i \log p_i \leqslant \log(n\times n(\gamma))\},\end{aligned}$$

where $n(\gamma)=\prod_i p_i^{\gamma_i}$. So since $T_q(n)\subseteq T_q(n\times n(\gamma))$, we deduce that

$$\frac{\#((T_q(n)+\gamma)\backslash T_q(n))}{\#T_q(n)} \leqslant \frac{\#T_q(n\times n(\gamma))-\#T_q(n)}{\#T_q(n)} \to_n 0 \text{ by } (5),$$

hence with (4), $(P2)$ is satisfied for the $\mathbb{N}^q$ case, and as a consequence of our study of the FD case we obtain

Corollary 1. ([6, Theorem 1]) *Any FD chain satisfies* $(P2)$, *whence is good.*

2.3. The inductive step for constructing a good ID chain. We set for $\gamma\in l_0(\mathbb{N})$ (resp. $\gamma\in\mathbb{N}^q$)

$$\partial_\gamma(T(n))=(T(n)+\gamma)\Delta T(n) \text{ (resp. } \partial_\gamma(T_q(n))=(T_q(n)+\gamma)\Delta T_q(n)).$$

We know by Lemma 1 that $(P2)$ is enough for an ID coprime generated chain to be good. And we also know by Corollary 1 that $(P2)$ holds in the FD case. The idea to reach the ID case is to show that given $p_1<\cdots<p_q$, it is possible to choose $p_{q+1}>p_q$ such that "small" increase occurs in the quotients $(P2)$ uniformly in γ belonging to some finite subset $\langle q\rangle$ of $l_0(\mathbb{N})$, where the increasing union over q of these subsets cover $l_0(\mathbb{N})$. This is done in the present section and summarized in Lemma 2 below.

If $q(n):=\max\{q : p_q\leqslant n\}$, then $T(n)=T_{q(n)}(n)$. Our argumentation shall strongly rely on this equality, on a careful use of (5) and the second estimate (cf. [5] where it is proved for the first q primes but carries out also in the case we need here)

$$\#((T_q(n)+\bar q)\backslash T_q(n)) \sim_{x\to\infty} \frac{\log(p_1^q\dots p_q^q)}{(q-1)!\prod_{i=1}^q \log p_i}(\log x)^{q-1}, \tag{6}$$

where $\bar q=(q,q,\dots,q)$.

We assume $q>1$ and that $p_1<\cdots<p_q$ are pairwise coprime. We define

$$\langle q\rangle := \{\gamma=(\gamma_i)\in\mathbb{N}^q : \gamma_i\leqslant q,\ 1\leqslant i\leqslant q\}.$$

Given arbitrary $\varepsilon_q>0$, by $(P2)$ for the FD case (Corollary 1), there exists an $N(\varepsilon_q)$ such that

$$x\geqslant N(\varepsilon_q) \Rightarrow \forall\gamma\in\langle q\rangle,\ \#\partial_\gamma(T_q(x))/\#T_q(x) < \frac{\varepsilon_q}{2}. \tag{7}$$

We now let $p_{q+1}>p_q$ denote an integer coprime to the previous numbers p_k, to be specified later on. We assume that $p_{q+1}\geqslant N(\varepsilon_q)$ ($N(\varepsilon_q)$ comes in (7)). Then if

$k \geqslant 1$ and $p_{q+1}^k \leqslant n < p_{q+1}^{k+1}$, we have

$$T_{q+1}(n) = \sum_{i=0}^{k} (\mathbb{N}^q \times \{i\}) \cap T_{q+1}(n) \ \text{ (a disjoint union).}$$

Let us put $T_{q+1}(n,i) := (\mathbb{N}^q \times \{i\}) \cap T_{q+1}(n),\ 0 \leqslant i \leqslant k = \left[\frac{\log n}{\log p_{q+1}}\right]$. Then we observe that $(0 \leqslant i \leqslant k)$

$$(\alpha_1, \ldots, \alpha_q, i) \in T_{q+1}(n,i) \Leftrightarrow (\alpha_1, \ldots, \alpha_q) \in T_q\left(\frac{n}{p_{q+1}^i}\right),$$

and moreover if $\gamma \in \langle q \rangle \subset \mathbb{N}^q$, then $\gamma_{q+1} = 0$, and therefore $\partial_\gamma(T_{q+1}(n)) = \sum_{i=0}^k \partial_\gamma(T_{q+1}(n,i))$ (disjoint union) where $\partial_\gamma(T_{q+1}(n,i)) = (T_{q+1}(n,i) + \gamma)\ \Delta T_{q+1}(n,i)$. Then for such γ,

$$(\alpha_1, \ldots, \alpha_q, i) + \gamma \in \partial_\gamma(T_{q+1}(n)) \Leftrightarrow (\alpha_1, \ldots, \alpha_q) + \gamma \in \partial_\gamma\left(T_q\left(\frac{n}{p_{q+1}^i}\right)\right).$$

We now define $(\gamma \leqslant \gamma') \Leftrightarrow (\forall i,\ \gamma_i \leqslant \gamma_i')$. An easy observation is

$$\gamma \leqslant \gamma' \Rightarrow \#\partial_\gamma(T_q(n)) \leqslant \#\partial_{\gamma'}(T_q(n)).$$

Hence with the above we get

$$\gamma \in \langle q \rangle \Longrightarrow \begin{cases} \#T_{q+1}(n) = \sum_{i=0}^k \#T_q\left(\frac{n}{p_{q+1}^i}\right), \\ \#\partial_\gamma(T_{q+1}(n)) \leqslant \#\partial_{\bar{q}}(T_{q+1}(n)) = \sum_{i=0}^k \#\partial_{\bar{q}}\left(T_q\left(\frac{n}{p_{q+1}^i}\right)\right). \end{cases}$$

Therefore as soon as $n, p_{q+1} \geqslant N(\varepsilon_q)$, if $k = \left[\frac{\log n}{\log p_{q+1}}\right]$, we have, using (7):

$$\begin{aligned} k = 0 \text{ (i.e. } N(\varepsilon_q) \leqslant n < p_{q+1}) &\Rightarrow T_{q+1}(n) = T_q(n) \\ &\Rightarrow \forall \gamma \in \langle q \rangle,\ \#\partial_\gamma(T_{q+1}(n))/\#T_{q+1}(n) < \tfrac{\varepsilon_q}{2}, \end{aligned}$$

and

$$k \neq 0 \Rightarrow \forall \gamma \in \langle q \rangle,$$

$$\begin{aligned} \#\partial_\gamma(T_{q+1}(n))/\#T_{q+1}(n) &\leqslant \#\partial_{\bar{q}}(T_{q+1}(n))/\#T_{q+1}(n) \\ &\leqslant \frac{\sum_{i=0}^{k-1} \#\partial_{\bar{q}}\left(T_q\left(\frac{n}{p_{q+1}^i}\right)\right)}{\sum_{i=0}^{k-1} \#T_q\left(\frac{n}{p_{q+1}^i}\right)} + \#\partial_{\bar{q}}\left(T_q\left(\frac{n}{p_{q+1}^k}\right)\right)/\#T_q\left(\frac{n}{p_{q+1}^{k-1}}\right) \\ &< \tfrac{\varepsilon_q}{2} + A\left(p_{q+1}, \frac{n}{p_{q+1}^k}\right), \end{aligned}$$

where $A(p_{q+1}, x) = \#\partial_{\bar{q}}(T_q(x))/T_q(p_{q+1}x)\ (x \geqslant 1)$.

Next we can write

$$A(p_{q+1}, x) = \frac{\#((T_q(x) + \bar{q}) \setminus T_q(x))}{\#T_q(p_{q+1}x)} + \frac{\#(T_q(x) \setminus (T_q(x) + \bar{q}))}{\#T_q(p_{q+1}x)}.$$

We firstly can estimate as in Section 2.2 that

$$\#(T_q(x) \setminus (T_q(x) + \bar{q})) \leqslant q \sum_{i=1}^{q} \#T_q^{(i)}(x) \leqslant q \sum_{i=1}^{q} \#T_q^{(i)}(p_{q+1}x),$$

which makes sure that, using (5), $\#(T_q(x)\backslash(T_q(x)+\bar{q}))/\#T_q(p_{q+1}x)\to 0$ as $p_{q+1}\to+\infty$, uniformly in $x\geqslant 1$.

Secondly, by $(5,6)$, there exist two positive constants C_1 and C_2, depending only on q, such that uniformly in $x\geqslant 1$ and p_{q+1},

$$\begin{cases} \#((T_q(x)+\bar{q})\backslash T_q(x))\leqslant C_1\log(x)^{q-1}, \text{ by } (6)\\ \#T_q(p_{q+1}x)\geqslant C_2\log(p_{q+1}x)^q, \text{ by } (5)\end{cases}$$

whence there exists some positive constant C depending only on $p_1,\dots,p_q$ such that uniformly in $x\geqslant 1$, for any p_{q+1},

$$\frac{\#((T_q(x)+\bar{q})\backslash T_q(x))}{\#T_q(p_{q+1}x)}\leqslant\frac{C}{\log p_{q+1}}.$$

Finally we may select $p_{q+1}\geqslant N(\varepsilon_q)$ so large that uniformly in $x\geqslant 1$,

$$A(p_{q+1},x)<\frac{1}{2}\varepsilon_q.$$

For such choice of p_{q+1}, we get that as soon as $n\geqslant N(\varepsilon_q)$, for any $\gamma\in\langle q\rangle$,

$$\#\partial_\gamma(T_{q+1}(n))/\#T_{q+1}(n)<\varepsilon_q, \tag{8}$$

we have proved:

Lemma 2. *Given $q>1$, arbitrary coprime $p_1<\cdots<p_q$, arbitrary $\varepsilon_q>0$, there exists an integer $N(\varepsilon_q)$ and a $p_{q+1}\geqslant N(\varepsilon_q)$ which is coprime to the p_i's $(1\leqslant i\leqslant q)$, such that for any $\gamma\in\langle q\rangle$, if $n\geqslant N(\varepsilon_q)$, (8) holds.*

2.4. The inductive construction of good ID chains. We fix a sequence $(\varepsilon_q)_{q\geqslant 1}$ of positive real numbers tending to 0. Next we select arbitrary $p_1>0$. Then a repeated inductive use of Lemma 2 produces a sequence $p_1<p_2<p_3<\cdots<p_{q+1}<\cdots$ of pairwise coprime integers, and another sequence $N(\varepsilon_1)\leqslant N(\varepsilon_2)\leqslant\cdots\leqslant N(\varepsilon_q)\leqslant\cdots$ of integers (we can choose them increasing), along with the corresponding properties in (8).

We then define, for each n, the set $T(n)$ as in (3). As before, $T(n)=T_{q(n)}(n)$, where $p_{q(n)}\leqslant n<p_{q(n)+1}$: then if $n>p_2$ (that is $q(n)\geqslant 2$), and $q\leqslant q(n)-1$,

$$\gamma\in\langle q\rangle\subset\langle q(n)-1\rangle\Rightarrow\#\partial_\gamma(T_{q(n)}(n))/\#T_{q(n)}(n)<\varepsilon_{q(n)-1},$$

because $p_{q(n)}$ exceeds $N(\varepsilon_{q(n)-1})$.

Now we fix $\gamma\in l_0(\mathbb{N})$ and select $q\geqslant 2$ such that $\gamma\in\langle q\rangle$. Then if n_0 satisfies $q(n_0)-1\geqslant q$, we obtain that for any $n\geqslant n_0$,

$$\#\partial_\gamma(T(n))/\#T(n)=\#\partial_\gamma(T_{q(n)}(n))/\#T_{q(n)}(n)<\varepsilon_{q(n)-1},$$

by our inductive construction using Lemma 2. Since $\varepsilon_q\to 0$ and $q(n)\to\infty$, this proves $(P2)$. By Lemma 1, we have proved Theorem 1.

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