Metric properties of generalized Cantor products

by

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Abstract. *Finite and absolutely continuous invariant measures for fibered generalized Cantor products (in the sense of Sierpinski) are described. The asymptotic behavior of the associated sequence of digits is studied. Lebesgue complete uniform distribution is proved for sequences associated in a natural way to these.*

0. Introduction.

Generalized Cantor products are algorithms that give a representation of real numbers $x \in [0,1[$ as infinite products of rational ones. They have been developed in [Opp] first. Let us present those we shall consider from the metric point of view in this paper.

The letter "k" shall denote an integer ≥ 1. For any $x \in [0,1[$, let $r_0(x) \in \mathbf{N}$ and $T(x) \in [0,1[$ be defined by

$$(1) \qquad \frac{r_0(x)-1}{r_0(x)+k-1} \leq x < \frac{r_0(x)}{r_0(x)+k}, \quad T(x) := x.\left(\frac{r_0(x)+k}{r_0(x)}\right).$$

One can see that $r_0(x) = \left[\frac{kx}{1-x}\right] + 1$. Define, for any real number $z \geq 1$,

$$(2) \qquad \begin{cases} a_z &= (z-1) \,/\, (z+k-1), \\ b_z &= a_z \,/\, a_{z+1} = a_{(z-1)(z+k)+1} \\ J_z &= [a_z,\ a_{z+1}[\,. \end{cases}$$

The sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are strictly increasing from 0 to 1. By definitions we have $\bigcup_{n\geq 1} J_n = [0,1[$, $J_n \cap J_m = \emptyset$ if $n \neq m$ and $T(x) = x.a_{n+1}^{-1}$ on J_n. Moreover

$$T\big(J_n\big) = \big[b_n,\ 1\big[.$$

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Thus, according to the terminology of F. Schweiger (see [Sch]), the triple $(T, [0,1[, (J_n)_{n\geq 1})$ is a measurable fibered system on $[0,1[$ with the Borel $\sigma-$algebra B.

Graph of T for $k = 2$.

Given $k \geq 1$ and $x \in [0,1[$, we define the sequence $\big(r_t(x)\big)_{t\geq 0}$ as follows :

$$r_t(x) = r_0(T^{(t)}(x)), \tag{3}$$

where $T^{(t)}$ denotes the t-th iterate of T ($T^{(0)} = Id_{[0,1[}$).

W. Sierpinski ([Sie-1]) and A. Oppenheim ([Opp]) showed that for any integer $k \geq 1$ and any $x \in [0,1[$, with (3),

$$x = \prod_{i=0}^{\infty} \frac{r_i(x)}{r_i(x) + k} \ . \tag{4}$$

The case $k = 1$ corresponds to the Cantor's product (see [Per]). In [Kn-Kn], generalizations of the Cantor's product that are given do not overlap with those from [Sie-1] or [Opp], and are not arising from fibered systems on $[0,1[$.

Euler's formula (see [MeFr-VdPo]) and Escott's formula ([Esc], [Sie-2])

$$\sqrt{\frac{x-1}{x+1}} = \prod_{n=0}^{+\infty} \left(\frac{\varphi^{(n)}(x)}{\varphi^{(n)}(x) + 1} \right), \quad \sqrt{\frac{x-2}{x+2}} = \prod_{n=0}^{+\infty} \left(\frac{\gamma^{(n)}(x-1)}{\gamma^{(n)}(x-1) + 2} \right),$$

where $\varphi(x) = 2x^2 - 1$ and $\gamma(z) = z^3 + 3z^2 - 2$, both give product expansions for integer x (with $k = 1$ or $k = 2$). Some other formulas can be derived from the work of Ostrowski [Ost] (see also [MeFr-VdPo]). P. Stambul ([Sta]) points out to us the following Cantor product expansion

$$\sqrt{2} - 1 = \prod_{n=0}^{+\infty} \left(\frac{\varphi^{(n)}(1)}{\varphi^{(n)}(1) + 1} \right),$$

where $\varphi(x) = 4x^2 - 1 + 2x\sqrt{2x^2 - 1}$ is not a polynomial. Thus, quadratic irrationals in $[0, 1[$ are not characterized by the fact that their sequence of digits for the Cantor product has ultimately polynomial growth (cf [Eng]).

In Part I we give some preliminary notations for cylinder sets and describe admissible sequences of digits $r_n(x)$ which occur in the product formula (4).

Our purpose is to study, as has been done for several other fibered systems (for instance continued fractions in [Khi]), the metric properties of the system $(T, [0, 1[, B)$. The motivation for this is that in the case of continued fractions, the asymptotic behavior for the relevant sequence of digits was deduced from the identification of the density $\frac{1}{\log 2.(1+x)}$ for a Lebesgue-continuous ergodic invariant measure on $[0, 1]$, for the transformation $x \mapsto \frac{1}{x} - \left[\frac{1}{x}\right]$ if $x \neq 0$, and $0 \mapsto 0$ (see [Khi], or [Sch]).

But it appears, in part 2, that the only probability invariant measure for T is the Dirac measure at 0, and that all σ–finite λ–continuous invariant measures for T are determined by their restrictions on wandering sets for T. Therefore, it should be the case that T is not ergodic with respect to λ.

However, in part 3, in analogy with what happens in the case of Sylvester's series (see [Ver], [Sch]), and in some sense quite at the opposite of what does for continued fractions, it appears that the limit function

$$\beta(x) = \lim_{n \to +\infty} \frac{\log\big(r_n(x)\big)}{2^n}$$

exists λ–a.e., and enables to conclude to the non ergodicity of T with respect to λ. The limit function β should be proved to have most of the properties the relevant one for Sylvester's series was proved to have in [Go-Sm], where it essentially was providing the first explicitly defined function having jointly continuous occupation density (see also [Gal]).

Finally, in part 4, we introduce the sequence of random variables $\big(t_n(.)\big)_{n \geq 0}$ defined on $[0, 1[$ by

$$t_n(x) = \frac{T^{(n+1)}(x) - b_{r_n(x)}}{1 - b_{r_n(x)}}, \; x \in [0, 1[, \; n \geq 0.$$

We show, using a modified version of a theorem of W. Philipp ([Phi]) in [Sch], chapter 11, that λ–a.e., the sequence $\big(t_n(x)\big)_{n\geq 0}$ is completely uniformly distributed modulo 1 (see [Ku-Ni]). This generalizes some similar uniform distribution for Sylvester's series, or Engel's series, proved in [Sch-1].

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1. Admissible sequences of digits

From [Sie-1] and the definition of T one has

$$x = \prod_{i=0}^{+\infty} \frac{r_i(x)}{r_i(x)+k}, \quad T^{n+1}(x) \in [b_{r_n}, 1[, \tag{5}$$

with $T^n(x) \in [\frac{r_n-1}{r_n+k-1}, \frac{r_n}{r_n+k}[$ and $r_n = r_n(x)$. This shall be called the $T-$expansion of x.

Take 1 as the value of the empty product, and let $n \geq 0$. One has

$$\begin{aligned} 0 < \prod_{j=0}^{n} \frac{r_j(x)}{r_j(x)+k} - x &< \Big(\prod_{j=0}^{n-1} \frac{r_j(x)}{r_j(x)+k}\Big)\Big(\frac{r_n(x)}{r_n(x)+k} - \frac{r_n(x)-1}{r_n(x)+k}\Big) \\ &< \frac{k}{(r_n(x)+k)(r_n(x)+k-1)}. \end{aligned}$$

Let n be an integer ≥ 1 and let $r := (r_0, \ldots,\ r_{n-1}) \in \mathbf{N}^{*n}$. The set

$$B(r) := J_{r_0} \cap T^{-1}\big(J_{r_1}\big) \cap \cdots \cap T^{-n+1}\big(J_{r_{n-1}}\big)\ .$$

is said to be a cylinder set of rank n if it is not empty. For $r = (r_0, \ldots, r_{n-1}) \in \mathbf{N}^n$ (respectively $p = (p_i)_{i\geq 0}$) and $j \in [0, n]$ (resp. $j \geq 0$), define

$$\mathbf{\Pi}_j(r) := \prod_{i=0}^{j-1} \frac{r_i}{r_i+k} \quad \left(\text{resp. } \mathbf{\Pi}_j(p) := \prod_{i=0}^{j-1} \frac{p_i}{p_i+k}\right). \tag{6}$$

If $B(r)$ is a cylinder set of rank n we easily get from (1), (2) and (5) :

$$B(r) = \big[\Pi_n(r).b_{r_{n-1}},\ \Pi_n(r)\big[\ . \tag{7}$$

Definition 1.1. *An $n-$uple $r = (r_0, \ldots, r_{n-1})$ (resp. a sequence $p = (p_m)_{m\geq 0} \in \mathbf{N}^{\mathbf{N}}$) is said to be a $T-$admissible $n-$uple (resp. sequence) of digits if $B(r) \neq \emptyset$ (resp. $B(p_0, \ldots, p_{n-1}) \neq \emptyset$ for all $n \geq 1$). The set of $T-$admissible $n-$uples will be denoted by A_n.*

From (5), p is a $T-$admissible sequence of digits if and only if for all $n \geq 0$, one has

$$[b_{p_n}, 1[\cap J_{p_{n+1}} \neq \emptyset.$$

Proposition 1.1. *A sequence $p = (p_n)_{n\geq 0}$ of natural numbers is a $T-$admissible sequence of digits if and only if for all $n \geq 0$ one has :*

$$p_{n+1} \geq p_n^2 + (p_n - 1)(k-1) \geq p_n^2.$$

Proof. Since b_r has the form $a_{(r-1)(r+k)+1}$, an admissible sequence $(p_n)_{n\geq 0}$ is characterized by the inequalities $b_{p_n} < a_{p_{n+1}+1}$, $n \geq 0$. In other words,

$$\frac{(p_n - 1)(p_n + k)}{(p_n - 1)(p_n + k) + k} < \frac{p_{n+1}}{p_{n+1} + k}.$$

After simplification, we get the desired inequality. ■

Remark 1.1. Let $p(.)$ be the polynomial $p(x) := x^2 + (x-1)(k-1)$. From (2) we have $a_n = a_{n+1}a_{p(n)} = a_{n+1}a_{p(n)+1}a_{p^2(n)}$. Hence by induction we obtain the following product formula

$$\frac{n-1}{n-1+k} = \prod_{j=1}^{\infty} \frac{p^{(j)}(n)}{p^{(j)}(n) + k}. \tag{8}$$

According to Proposition 1.1,, formula (8) gives the $T-$expansion of $\frac{n-1}{n-1+k}$, for $n \in \mathbf{N}$ (this was known from [Opp]). However formula (8) holds for all real numbers $k \geq 1$ and $n \geq 1$.

2. Invariant measures

The transformation T is such that $T(0) = 0$ and if $x \in]0, 1[$, the sequence $(T^{(n)}(x))_{n\geq 0}$ is strictly increasing to 1. Thus, from the Riesz representation theorem and the individual ergodic theorem, using Cesaro means, taking any generic point for μ if μ is an ergodic invariant probability measure, one can see that necessarily, for any $f \in \mathcal{C}([0, 1[)$, $\int f d\mu =$

$\lim_{x\to 1^-} f(x)$: since $T(0) = 0$ is the only fixed point for T, one must have $\mu = \delta_0$, where δ_0 denotes the Dirac measure at point 0.

Remark 2.1. It is more interesting to consider probability measures μ which are quasi-invariant under T, that is to say μ is equivalent to $\mu \circ T^{-1}$. We give an example of such a measure which is discrete. Let β_j, $j \in \mathbf{Z}$ be the points in [0,1[(identified to $\mathbf{X}$) given by

$$\beta_n := \frac{p^{(n)}(2) - 1}{p^{(n)}(2) - 1 + k} \quad \text{and} \quad \beta_{-n} = (k+1)^{-n-1}$$

for $n = 0, 1, 2, \ldots$. By (5) and (8) one has

$$T^n\left(\frac{1}{k+1}\right) = \prod_{j=0}^{\infty} \frac{p^{(j)}(p^{(n)}(2))}{p^{(j)}(p^{(n)}(2)) + k}$$

for $n \geq 0$ and $T((k+1)^{-(m+1)}) = (k+1)^{-m}$ for $m \geq 1$. Hence $T(\beta_n) = \beta_{n+1}$ for all $n \in \mathbf{Z}$. Let δ_a denote the Dirac measure at a then $\delta_{b_n} \circ T^{-1} = \delta_{b_{n+1}}$. This proves that the probability measure $\mu := \dfrac{1}{3} \sum\limits_{n \in \mathbf{Z}} 2^{-|n|} \delta_{\beta_n}$ is quasi-invariant under T.

Now let us look at σ–finite λ–continuous invariant measures. Let U be any proper neighbourhood of 1, e.g. take $U = [a, 1]$, $0 < a < 1$, and extend T from $[0, 1[$ to the 1-torus $[0, 1]$ setting $T(1) = 1 = 0$. Let $V = T^{-1}(U) \setminus U$. Then define $V_n = T^n(V)$, $n \in \mathbf{Z}$. It is a so called wandering set ; indeed, using the fact that the sequence $(T^n(U))_{n \in \mathbf{Z}}$ is decreasing, one has

$$\bigcup_{n=-\infty}^{+\infty} V_n = [0, 1], \;\; \text{and} \;\; V_n \cap V_m = \emptyset \;\; \text{for} \;\; m \neq n. \tag{9}$$

Now assume we want to determine the density for a σ–finite T–invariant λ–continuous measure. Then if we take any positive, measurable and σ–finite function on V, we can define it on any V_n, taking its image via T^n, and finally we obtain a σ–finite density for a T–invariant λ–continuous measure (use (9)). For example, take $a = \frac{k+2}{2(k+1)}$; then $V = \left[\frac{k+2}{2(k+1)^2}, \frac{k+2}{2(k+1)}\right[$.

3. Non ergodicity of T with respect to λ, and asymptotic behavior of $\big(r_n(x)\big)_{n\geq 0}$.

Lemma 3.1. *There are two positive constants d_1 and d_2 such that for any non empty cylinder set $B(r_0,\dots,r_{n-1})$ of rank $n\geq 1$ and for any integers w, j, $(w\geq j\geq 1)$, such that $B(r_0,\dots,r_{n-1},j,w)$ is a non empty cylinder set of rank $n+2$ one has*

$$d_1\cdot\frac{j^2}{w^2}\leq\frac{\lambda\big(B(r_0,\dots,r_{n-1},j,w)\big)}{\lambda\big(B(r_0,\dots,r_{n-1},j)\big)}\leq d_2\cdot\frac{j^2}{w^2}\ .$$

Proof. Put $B=B(r_0,\dots,r_{n-1},j,w)$, $A=B(r_0,\dots,r_{n-1},j)$ and $P=\Pi_n(r)$ for short (see (6)), where $r=(r_0,\dots,r_{n-1})$ (cf. (6)). Then, with (7),

$$\lambda(A)=P\cdot\frac{k}{(j+k)(j+k-1)},\quad\text{and}\quad\lambda(B)=P\cdot\frac{jk}{(j+k)(w+k)(w+k-1)}.$$

Therefore,

$$\frac{\lambda(B)}{\lambda(A)}=\frac{j(j+k-1)}{(w+k)(w+k-1)},$$

and the inequalities of the Lemma follow with constants (for example) $d_1=(k^2+k)^{-1}$ and $d_2=k$. ■

Lemma 3.2. *The limit function $\beta(x):=\lim_{n\to\infty}\frac{\log(r_n(x))}{2^n}$ exists $\lambda-$a.e. Moreover, $\beta(.)$ is measurable and there exists a constant $\gamma>0$ such that for all $j\geq 1$, $n\geq 0$ and all $\varepsilon>0$ one has*

$$(10)\qquad\begin{cases}\lambda(\{x;\ r_n(x)=j\text{ and }0\leq\beta(x)-2^{-n}\log j\leq\varepsilon\})\geq\Big(1-\frac{2}{e^{\gamma\varepsilon 2^n}-1}\Big)\lambda(\{r_n=j\}),\\ \text{and }\lambda\text{−a.e.,}\\ \beta(x)=\frac{1}{2}\left(\log r_1(x)+\displaystyle\sum_{n=0}^{+\infty}\frac{\log\big(\frac{r_{n+1}(x)}{r_n(x)^2}\big)}{2^n}\right).\end{cases}$$

Proof. The second part of formula (10) is obvious, when the λ–a.e existence of the limit function β is known.

Let $\varepsilon>0$ and for $x\in[0,1[$ define $\beta_n(x):=2^{-n}\log(r_n(x))$. Since $r_{n+1}(x)\geq r_n(x)^2$, the sequence $\big(\beta_n(x)\big)_{n\geq 0}$ is not decreasing. Then $\beta_{n+1}(x)-\beta_n(x)>\varepsilon$ is equivalent to $r_{n+1}(x)>exp(\varepsilon.2^{n+1})r_n(x)^2$. From Lemma 3.1, we get

$$(11)\qquad\lambda\{r_n=j\ \text{ and }\ \beta_{n+1}-\beta_n>\varepsilon\}\leq d_2\Big(\sum_{\substack{w,\\ w>j^2\exp(\varepsilon 2^{n+1})}}\frac{j^2}{w^2}\Big)\lambda\{r_n=j\}.$$

But it follows from elementary calculus that for all $j \geq 1$,

$$\sum_{\substack{w,\\ w > j^2 \exp(\varepsilon 2^{n+1})}} \frac{j^2}{w^2} \leq \frac{2}{e^{\varepsilon 2^{n+1}}} \tag{12}$$

Using (11) and (12), we obtain

$$\lambda(\{r_n = j \text{ and } \beta_{n+1} - \beta_n > \varepsilon\}) \leq 2e^{-\varepsilon 2^{n+1}} \lambda(\{r_n = j\}).$$

Define $\eta_m = (\sqrt{2} - 1)(\sqrt{2})^{-(m+1)}$, such that $\sum_{m \geq 1} \eta_m = 1$. Let $n \geq 0$, $m \geq 1$ be integers and assume $\beta_{n+s}(x) - \beta_{n+s-1}(x) \leq \varepsilon \eta_s$ for all $s \in \{1, 2, \dots, m\}$. Then $\beta_{n+m}(x) - \beta_n(x) \leq \varepsilon$ so that for

$$X_n(j; \varepsilon) := \{x;\ r_n(x) = j \text{ and } \exists\, m \geq 1,\ \beta_{n+m}(x) - \beta_n(x) > \varepsilon\}$$

we obtain

$$\begin{aligned}
\lambda(X_n(j;\varepsilon)) &\leq \lambda(\{r_n = j \text{ and } \exists m \geq 1,\ \beta_{n+m} - \beta_{n+m-1} > \varepsilon \eta_m\}) \\
&\leq 2\Big(\sum_{m \geq 1} e^{-\varepsilon \eta_m 2^{n+m+1}}\Big) \lambda(\{r_n = j\}) \\
&\leq \frac{2}{e^{\gamma \varepsilon 2^n} - 1} \lambda(\{r_n = j\}) \qquad (13)
\end{aligned}$$

where $\gamma = \sqrt{2} - 1$. But (13) is nothing that inequality (10) of Lemma 3.2. If we sum over j all inequalities (10) (n fixed) we also get

$$\lambda(\{\beta - \beta_n \leq \varepsilon\}) \geq 1 - \frac{2}{e^{\gamma \varepsilon 2^n} - 1}.$$

Now it is quite clear that the sequence $(\beta_n(x))_{n \geq 0}$ converges (in $[0, +\infty[$) for almost all $x \in [0, 1[$. Since β_n is measurable, β also is. ■

Remark 3.1. Notice that β satisfies the following functional equations :

$$\beta(Tx) = 2\beta(x) \quad \text{and} \quad \beta\big(\frac{1}{k+1} x\big) = \frac{1}{2}\beta(x).$$

As in the case of Sylvester's series (see [Go-Sm]), it can be proved that β is dense in its epigraph and has local minimas at rational points exactly. In [Go-Sm] was first proved that the β function for Sylvester's series has a $\mathcal{C}^\infty$ density. In [Gal], it was proved that for the Cantor product, β has a $\mathcal{C}^1$ density. This last result at least should hold for the generalized Cantor products we are dealing with here.

Theorem 3.1. *T is not ergodic with respect to λ, i.e. there exists two disjoint T–invariant subsets of $[0,1[$ with positive Lebesgue measure.*

Proof of Theorem 3.1. Let J be a non empty open sub-interval of $]0,\infty[$. Choose $\varepsilon > 0$ such that there exist integers $p \geq 1$ and $m \geq 1$, satisfying

$$[\frac{\log(p)}{2^m} - \varepsilon, \frac{\log(p)}{2^m} + \varepsilon] \subset J.$$

Let N_ε be an integer such that $1 - \frac{2}{e^{\gamma\varepsilon 2^n}-1} > 0$ for all $n \geq N_\varepsilon$. We can easily choose integers $d \geq 2$ and $n \geq N_\varepsilon$ in order to have $2^{-n}\log d$ close enough to $2^{-m}\log p$ such that we still have

$$[\frac{\log(d)}{2^n} - \varepsilon, \frac{\log(d)}{2^n} + \varepsilon] \subset J.$$

Since $\lambda(\{r_n = d\}) > 0$ for any integer $d \geq 1$, inequality (10) implies $\lambda(\{x;\ \beta(x) \in J\}) > 0$ and the set

$$E(J) := \{x;\ \beta(x) \in \bigcup_{m\in\mathbf{Z}} 2^m J\}$$

is measurable and $T-$invariant with $\lambda(E(J)) > 0$. Let J and J' be two non empty open intervals such that $J \subset [\frac{1}{2}, \frac{3}{4}[$ and $J' \subset [\frac{3}{4}, 1[$. Then the sets $E(J)$ and $E(J')$ are disjoint, $T-$invariant and

$$\mu(E(J)) > 0 \quad \text{and} \quad \mu(E(J')) > 0.$$

This ends the proof. ■

4. Uniform distribution

In this section we study the distribution of $T^n(x)$ in the interval $[a_{r_n(x)}, a_{r_n(x)+1}[$. More precisely let $(t_n(.))_{n\geq 0}$ be the sequence of random variables defined on $[0,1[$ by

$$\begin{aligned} t_n(x) &:= \frac{T^n(x) - a_{r_n}}{a_{r_n+1} - a_{r_n}} \\ &= \frac{T^{n+1}(x) - b_{r_n(x)}}{1 - b_{r_n(x)}}, \quad x \in [0,1[, \quad n \geq 0. \end{aligned}$$

Let $\Phi_n(.)$ denote the distribution function of $t_n(.)$, and define

$$W_n(d) := \{x;\ 0 \leq t_n(x) < d\}\ ,\ d \in [0,1].$$

Theorem 4.1. *The sequence of random variables $(t_n(.))_{n\geq 0}$ is identically and uniformly distributed (i.e., $\Phi_n(d) = d$ for $0 \leq d \leq 1,\ n \geq 0$).*

Proof. For $d \in [0,1]$ we have $\Phi_n(d) = \lambda(\{\ x;\ 0 \leq t_n(x) < d\ \})$. Let $r = (r_0, \dots, r_n) \in A_{n+1}$ (see Definition 1.1). Since $T^{n+1}(x) = \Pi_{n+1}^{-1}(r).x$ on $B(r_0, \dots, r_n)$ and $T^{n+1}(B(r)) = [b_{r_n}, 1[$, the set $W_n(d)$ is the union of the following pairwise disjoint sets

$$B(r) \cap W_n(d) = \{x;\ b_{r_n}\Pi_{n+1}(r) \leq x < \Pi_{n+1}(r)(b_{r_n} + d(1 - b_{r_n}))\}.$$

But $\lambda(B(r) \cap W_n(d)) = d\lambda(B(r))$ so

$$\lambda(W_n(d)) = \sum_{r \in A_{n+1}} d\lambda(B(r)) = d. \qquad \blacksquare$$

With in mind the study of the $\lambda-$a .e. complete uniform distribution of the sequence $\big(t_n(x)\big)_{n\geq 0}$, let us introduce the following ;

Definition 4.1. *Let $p \in \mathbf{N}$ and $(d_0, \dots, d_p),\ (d'_0, \dots, d'_p) \in [0,1]^{p+1}$. Then, for any $n \geq 0$, let $E_n(d_0, \dots, d_p) = W_n(d_0) \cap \dots \cap W_{n+p}(d_p)$. If $m \geq 1$, let*

$$(d_0, \dots, d_p, 1^m, d'_0, \dots, d'_p) = (d_0, \dots, d_p, \underbrace{1, \dots, 1}_{m \text{ times}}, d'_0, \dots, d'_p).$$

Let $d_{-1} = 1$ and $E_n(\emptyset) = [0,1]$.

With the above notations, we have ;

Theorem 4.2. *For any integer $p \geq 0$, for any integer $n \geq 1$, any integer $m \geq 0$, any $(d_0, \dots, d_p, d'_0, \dots, d'_p) \in [0,1]^{2(p+1)}$,*

$$(\alpha)\quad \big|\lambda\big(E_n(d_0, \dots, d_p, 1^m, d'_0, \dots, d'_p)\big) - d_0 \cdots d_p d'_0 \cdots d'_p\big| \leq 20(p+1)^2 k^2 (k+1)^2 \left(\frac{1}{2}\right)^n,$$

and

$$(\beta)\qquad \big|\lambda\big(E_n(d_0, 1^m, d'_0)\big) - d_0 d'_0\big| \leq \frac{5}{2} k^2 (k+1)^2 \left(\frac{1}{2}\right)^{n+m}.$$

Proof : STEP 1. We need several lemmas and definitions ;

Lemma 4.1. *For any $n \in \mathbf{N}$, $m \geq 1$, $r = (r_0, r_1, \ldots, r_{n+m}) \in A_{n+m+1}$, one has*

$$\frac{r_n^2 \lambda(B(r_{n+1}, \ldots, r_{n+m}))}{k} \leq \frac{\lambda(B(r))}{\lambda(B(r_0, \ldots, r_n))} \leq (k+1) r_n^2 \lambda(B(r_{n+1}, \ldots, r_{n+m})) \tag{14}$$

$$\leq \frac{k(k+1)}{2^m}. \tag{15}$$

Moreover

$$\lambda(B(r_0, \ldots, r_n)) \leq \min\left\{2^{-(n+1)}, \frac{k}{(r_n + k)(r_n + k - 1)}\right\}. \tag{16}$$

Proof. Notice that

$$\lambda(B(r)) = \left(\frac{r_0}{r_0 + k} \cdots \frac{r_{n+m-1}}{r_{n+m-1} + k}\right) \frac{k}{(r_{n+m} + k)(r_{n+m} + k - 1)}$$

$$= \lambda(B(r_0, \ldots, r_n)) \frac{(r_n + k)(r_n + k - 1)}{k} \lambda(B(r_{n+1} \ldots r_{n+m}))$$

and then inequality (14) follows from $\frac{x^2}{k} \leq \frac{(x+k)(x+k-1)}{k} \leq (k+1)x^2$, for $x \geq 1$. On the other hand, put $p(x) = x^2 + (x-1)(k-1)$ and assume that $r_{s-1} = 1 (\neq r_s)$ for a digit with $0 < s \leq n$. Proposition 1.1 and (7) imply

$$\lambda(B(r_0, \ldots, r_n)) \leq (k+1)^{-s} \frac{k}{(p^{(n-s)}(r_s) + k - 1)(p^{(n-s)}(r_s) + k)}.$$

If $r_s = 1 = r_n$ the inequality (15) is evident. Otherwise $r_s \geq 2$ but $p^{(n-s)}(2) \geq 2^{2^{n-s}}$ therefore (16) is still true. It remains to prove (15). If $r_n = 1$, the inequality follows from (16), otherwise we have

$$\lambda(B(r_{n+1}, \ldots, r_{n+m})) \leq k (p^{(m)}(r_n))^{-2} \leq k r_n^{-2^{m+1}} \leq k r_n^{-2} 2^{-m}. \qquad \blacksquare$$

Lemma 4.2. *For positive natural numbers n and m let*

$$F_n(m) = \#\{\, (r_0, \ldots, r_{n-2}) \in \mathbf{N}^{n-1},\ (r_0, \ldots, r_{n-2}, m) \in A_n \,\}.$$

Then $F_n(m) \leq m$.

Proof. We use induction on n. It is clear that $F_1(m) \leq m$. Now, let $n \geq 1$ be given and assume $F_n(m) \leq m$ for all $m \geq 1$. Proposition 1.1 implies that for any $(r_0, \ldots, r_{n-1}, m) \in A_{n+1}$ one has $r_{n-1} \leq \sqrt{m}$. Therefore

$$F_{n+1}(m) \leq \sum_{1 \leq j \leq \sqrt{m}} j \leq m. \qquad \blacksquare$$

Lemma 4.3. *For any positive natural numbers n, m and for any map $s : A_n \to \mathbf{N}^m$ satisfying $((r_0,\dots,r_{n-1}), s(r_0,\dots,r_{n-1})) \in A_{n+m}$, one has*

$$\sum_{r\in A_n} \lambda(B(r,s(r))) \le \frac{5k^3(k+1)^3}{2^{n+m}}$$

(we identify $\mathbf{N}^{n+m}$ with $\mathbf{N}^n \times \mathbf{N}^m$).

Proof. We first study the case $m = 1$. If $n = 1$, first notice that for any application $s_1 : A_1 = \mathbf{N}^* \to \mathbf{N}^*$ such that for any $r \in \mathbf{N}^*$, $(r, s_1(r)) \in A_2$, from (7) and Proposition 1.1,

$$\sum_{r\in\mathbf{N}^*} \lambda\big(B(r,s_1(r))\big) \le \sum_{r\ge 1} \frac{kr}{(r+k)(s_1(r)+k)(s_1(r)+k-1)}$$

$$\le \sum_{r\ge 1} \frac{kr}{(r+k)(r^2+(k-1)r+1)(r^2+(k-1)r)}.$$

But since $k \ge 1$,

$$\sum_{r\ge 1} \frac{k}{(r+k)(r^2+(k-1)r+1)(r+k-1)} \le \sum_{r\ge 1} \frac{1}{(r+1)(r^2+1)} \le \frac{1}{2},$$

and indeed $2 \le 5k^3(k+1)^3$.

Assume now that $n \ge 2$. Then from Lemma 4.1, it follows that for any $r \in A_n$, $r = (r_0,\dots,r_{n-1})$,

$$\lambda\big(B(r,s(r))\big) \le k(k+1)\lambda(B(r))\frac{r_{n-1}^2}{p(r_{n-1})^2} \le k(k+1)\frac{\lambda\big(B(r)\big)}{r_{n-1}^2} \le \frac{k^2(k+1)}{r_{n-1}^4}.$$

Then, for any $N \ge 1$,

$$\sum_{r\in A_n} \lambda\big(B(r,s(r))\big) \le k^2(k+1) \sum_{\substack{r\in A_n,\\ r_{n-1}>N}} \frac{1}{r_{n-1}^4} + \sum_{\substack{r\in A_n,\\ r_{n-1}\le N}} \lambda\big(B(r,s(r))\big)$$

$$\le k^2(k+1)\sum_{t>N}\frac{1}{t^3} + k(k+1) \sum_{\substack{r\in A_n,\\ r_{n-1}\le N}} \frac{\lambda(B(r))}{r_{n-1}^2}$$

$$\le \frac{k^2(k+1)}{2N^2} + k^2(k+1)^2 \sum_{\substack{(r_0,\dots,r_{n-1})\in A_n,\\ r_{n-1}\le N}} \lambda(B(r_0,\dots,r_{n-2}))\frac{r_{n-2}^2}{r_{n-1}^4}.$$

But $r_{n-1} \geq r_{n-2}^2$ therefore with $g = 4k^2(k+1)^2$ and (16),

$$\sum_{r \in A_n} \lambda(B(r, s(r)) \leq \frac{k^2(k+1)}{2N^2} + \frac{g}{2^{n+1}} \sum_{\substack{(r_0, \ldots, r_{n-2}) \in A_{n-1}, \\ r_{n-2} \leq \sqrt{N}}} r_{n-2}^{-6}$$

$$\leq \frac{k^2(k+1)}{2N^2} + \frac{g}{2^{n+1}} \sum_{1 \leq k \leq \sqrt{N}} k^{-5}$$

Passing to the limit as N tends to infinity, we get the case $m = 1$ with $\frac{5}{4}g$. The general case follows from (15) which gives $\lambda\left(B(r, s(r))\right) \leq \lambda\left(B(r, s_1(r))\right) \frac{k(k+1)}{2^{m-1}}$. ■

Definition 4.2. *Let $n \geq 1$ be an integer. Let $r = (r_0, \ldots, r_{n-1}) \in A_n$. Let $d \in [0, 1[$. Then define $r'(d, r)$ to be the unique integer such that, if $r'' = \big(r_0, \ldots, r_{n-1}, r'(d, r)\big)$, we have*

$$\Pi_n(r).\big(b_{r_{n-1}} + d(1 - b_{r_{n-1}})\big) \in B(r'').$$

Denote the above admissible $(n+1)-$uple r'' by $rr'(d, r)$ (as a concatenation). If $(r, r') \in \mathbf{N}^n \times \mathbf{N}^m$, let rr' be the $(n+m)-$uple defined by $rr' = (r_0, \ldots, r_{n-1}, r'_0, \ldots, r'_{m-1})$. Endow the sets A_n with the lexicographic order. If $d = 1$, and $r \in A_n$, let $r'(1, r) = +\infty$, and $B(r + \infty) = \emptyset$.

Let $n \geq 0$ and $m \geq 1$. Let $r \in A_{n+1}$, $r = (r_0, \ldots, r_n)$, and define

$$A_{n+1,m}(r) := \{\ r' = (r'_{n+1}, \ldots, r'_{n+m}) \in \mathbf{N}^m,\ rr' \in A_{n+m+1}\ \}.$$

Lemma 4.4. *For any $q \geq 1$, and any $k \geq 1$,*

$$\frac{1}{(q+k).(q+k-1)} >$$

$$2\left(\sum_{m \geq 0} \frac{1}{(q+m+k)\big((q+m)^2 + (q+m)(k-1) + 1\big)(q+m+k-1)}\right).$$

Proof. The sum of the series is clearly bounded by

$$\frac{1}{(q+k)(q+k-1)(q^2 + q(k-1) + 1)} +$$

$$\left(\sum_{t\geq q+1}\frac{1}{(t+k)(t+k-1)}\right)\frac{1}{(q+1)^2+(q+1)(k-1)+1}$$
$$\leq \frac{1}{(q+k)(q+k-1)}\left(\frac{1}{(q+1)(q+k)-2q-k+1}+\frac{1}{q+1}-\frac{1}{(q+1)(q+k)}\right)$$
$$\leq \left(\frac{1}{q+1}\right)\frac{1}{(q+k)(q+k-1)},$$

and $q\geq 1$. ■

STEP 2. Proof of Theorem 4.2. Let $p'\geq 1$. Using refining partitions of cylinders on $[0,1[$, one can see quite easily, with the use of Theorem 4.1. and Definition 4.2., that, given $(d_0,\ldots,d_{p'})\in[0,1]^{p'+1}$, $n\geq 1$ and $r=(r_0,\ldots,r_n)\in A_{n+1}$,

(17) $\lambda\big(E_n(d_0,\ldots,d_{p'})\cap B(r)\big)=$

$$\sum_{\substack{r_{n+1}\in A_{n+1,1}(r),\\ r_{n+1}<r'(d_0,r)}}\Bigg(\sum_{\substack{r_{n+2}\in A_{n+2,1}(rr_{n+1}),\\ r_{n+2}<r'(d_1,rr_{n+1})}}\cdots\Bigg(\sum_{\substack{r_{n+p'}\in A_{n+p',1}(rr_{n+1}\cdots r_{n+p'-1}),\\ r_{n+p'}<r'(d_{p'-1},rr_{n+1}\cdots r_{n+p'-1})}} d_{p'}.\lambda\Big(B(rr_{n+1}\ldots r_{n+p'})\Big)\Bigg)\Bigg)$$
$$+\sum_{\substack{r_{n+1}\in A_{n+1,1}(r),\\ r_{n+1}<r'(d_0,r)}}\Bigg(\cdots\Bigg(\sum_{\substack{r_{n+p'-1}\in A_{n+p'-1,1}(r..r_{n+p'-2}),\\ r_{n+p'-1}<r'(d_{p'-2},r..r_{n+p'-2})}}\lambda\Big(B\Big(r..r_{n+p'-1}r'(d_{p'-1},r..r_{n+p'-1})\Big)\cap E_n(d_0,..,d_{p'})\Big)\Bigg)\Bigg)$$
$$+\cdots\cdots$$
$$+\sum_{\substack{r_{n+1}\in A_{n+1,1}(r),\\ r_{n+1}<r'(d_0,r)}}\lambda\Big(B\Big(rr_{n+1}r'(d_1,rr_{n+1})\Big)\cap E_n(d_0,\ldots,d_{p'})\Big)$$
$$+\lambda\Big(B\Big(rr'(d_0,r)\Big)\cap E_n(d_0,\ldots,d_{p'})\Big).$$

Let, for $i\in[1,p]$,

(18) $$X_i(d_0,\ldots,d_p,n)=\left|\lambda\left(E_n(d_0,\ldots,d_i)\right)-d_i\lambda\left(E_n(d_0,\ldots,d_{i-1})\right)\right|.$$

Notice that $X_i(d_0,\ldots,d_p,n)=0$ if $p=0$ or $d_i\in\{0,1\}$. Let, for $i\in[1,p]$,

(19) $$Y_i(d_0,\ldots,d_p,n)=$$
$$\sum_{r\in A_{n+1}}\Bigg(\cdots\Bigg(\sum_{\substack{r_{n+i}\in A_{n+i,1}(rr_{n+1}\cdots r_{n+i-1}),\\ r_{n+i}<r'(d_{i-1},rr_{n+1}\cdots r_{n+i-1})}}\lambda\Big(B\Big(rr_{n+1}\ldots r_{n+i}r'(d_i,r\ldots r_{n+i})\Big)\cap E_n(d_0,\ldots,d_p)\Big)\Bigg)\cdots\Bigg),$$

and

(19) − bis $$Y_0(d_0,\ldots,d_p,n)=\sum_{r\in A_{n+1}}\lambda\left(B\big(rr'(d_0,r)\big)\cap E_n(d_0,\ldots,d_p)\right).$$

Definition 4.3. *Let $r'(r)$ denote the smallest element of $A_{n,1}(r)$ for $r \in A_n$.*

Let, for $i \in \mathbf{N}^*$, with Definitions 4.2 and 4.3,

$$(20) \qquad R_i(n) = \sum_{r \in A_{n+1}} \Big(\cdots \Big(\sum_{r_{n+i} \in A_{n+i,1}(rr_{n+1} \dots r_{n+i-1})} \lambda \Big(B \Big(rr_{n+1} \dots r_{n+i} r'(r \dots r_{n+i}) \Big) \Big) \Big) \cdots \Big),$$

and

$$(20) - \text{bis} \qquad R_0(n) = \sum_{r \in A_{n+1}} \lambda \left(B \big(rr'(d_0, r) \big) \right).$$

Define, for $i \in [1, p]$,

$$(21) \quad Z_i(d_0, \dots, d_p, n) = \sum_{r \in A_{n+1}} \Big(\cdots \Big(\sum_{\substack{r_{n+i} \in A_{n+i,1}(rr_n \cdots r_{n+i-1}), \\ r_{n+i} < r'(d_{i-1}, rr_n \cdots r_{n+i-1})}} \lambda \Big(B \Big(rr_n \dots r_{n+i} r'(d_i, r \dots r_{n+i}) \Big) \Big) \Big) \cdots \Big),$$

and

$$(21) - \text{bis} \qquad Z_0(d_0, \dots, d_p) = \sum_{r \in A_{n+1}} \lambda \left(B \big(rr'(d_0, r) \big) \right).$$

Observe that if $p > 0$,

$$(22) \quad \Bigg| \sum_{r \in A_{n+1}} \Big(\sum_{\substack{r_{n+1} \in A_{n+1,1}(r), \\ r_{n+1} < r'(d_0, r)}} \Big(\cdots \sum_{\substack{r_{n+p} \in A_{n+p,1}(rr_{n+1} \cdots r_{n+p-1}), \\ r_{n+p} < r'(d_{p-1}, rr_{n+1} \cdots r_{n+p-1})}} \lambda \Big(B(rr_{n+1} \dots r_{n+p}) \Big) \Big) \Big)$$

$$-d_{p-1} \left(\sum_{r \in A_{n+1}} \Big(\sum_{\substack{r_{n+1} \in A_{n+1,1}(r), \\ r_{n+1} < r'(d_0, r)}} \Big(\cdots \sum_{\substack{r_{n+p-1} \in A_{n+p-1,1}(rr_{n+1} \cdots r_{n+p-2}), \\ r_{n+p} < r'(d_{p-2}, rr_{n+1} \cdots r_{n+p-2})}} \lambda \Big(B(rr_{n+1} \dots r_{n+p-1}) \Big) \Big) \Big) \right) \Bigg|$$

$$\leq Z_{p-1}(d_0, \dots, d_p, n).$$

Then, from relations (17) to (22), if we put $Z_{-1}(d_0, n) = 0$,

$$(23) \qquad |\lambda (E_n(d_0, \dots, d_p)) - d_p \lambda (E_n(d_0, \dots, d_{p-1}))|$$

$$\leq \delta_{p \neq 0} . \delta_{d_p \notin \{0,1\}} . \left(2 \left(\sum_{i=0}^{p-1} Y_i(d_0, \dots, d_p, n) \right) + Z_{p-1}(d_0, \dots, d_p, n) \right)$$

$$\leq \underbrace{\delta_{p\neq 0}.\delta_{d_p\notin\{0,1\}}.\left(2\left(\sum_{i=0}^{p-1} R_i(n)\right)+Z_{p-1}(d_0,\ldots,d_p,n)\right)}_{W(d_0,\ldots,d_p,n)},$$

where if P is a proposition, $\delta_P = 0$ if P is false, 1 otherwise. Let $(d_0,\ldots,d_p,1^m,d'_0,\ldots,d'_p) = (a_0,\ldots,a_{2p+m+1})$. From (17), (18), Definition 4.2 and repeated application of the triangle inequality,

$$(24)\quad \left|\lambda\left(E_n(d_0,\ldots,d_p,1^m,d'_0,\ldots,d'_p)\right) - d_0\cdots d_p d'_0\cdots d'_p\right|$$

$$\leq \sum_{i=1}^{p} X_i(d_0,\ldots,d_p,n) + \sum_{i=p+m+1}^{2p+m+1} X_i(a_0,\ldots,a_{2p+m+1},n).$$

From Proposition I.1, Definition 4.3, for any integer $m \geq 1$, for any $r \in A_m$,

$$\sum_{p\geq r'(r)} \lambda\Big(B\big(rpr'(rp)\big)\Big) \leq \sum_{p\geq r'(r)} \left(\prod_{i=0}^{m-1} \frac{r_i}{r_i+k}\right) \frac{kp}{(p+k)(p^2+(k-1)p+1)(p^2+(k-1)p)},$$

and from Lemma 5.4, with $q = r'(r)$, we deduce from the above inequality that

$$\sum_{p\geq r'(r)} \lambda\left(B\big(rpr'(rp)\big)\right) \leq \left(\frac{1}{2}\right)\lambda\left(B\big(rr'(r)\big)\right).$$

Then, from definitions (20), (20)–bis and the above,

$$(25)\qquad R_i(n) \leq \left(\frac{1}{2}\right) R_{i-1}(n) \leq \cdots \leq \left(\frac{1}{2}\right)^i R_0(n).$$

It follows from (20), (21), (23), (24), and (25), that

$$(26)\quad \left|\lambda\left(E_n(d_0,\ldots,d_p,1^m,d'_0,\ldots,d'_p)\right) - d_0\cdots d_p d'_0\cdots d'_p\right|$$

$$\leq \sum_{i=1}^{p} W(d_0,\ldots,d_i,n) + \sum_{i=p+m+1}^{2p+m+1} W(a_0,\ldots,a_i,n)$$

$$\leq 4p(p+1)R_0(n) + 2(p+1)^2 R_{p+m+1}(n).$$

From Lemma 4.3, we have

$$(27)\qquad R_0(n) \leq \frac{5k^2(k+1)^2}{2^{n+1}}.$$

Thus, from (25), (26), (27), we obtain

$$(28) \quad \left|\lambda\left(E_n(d_0,\ldots,d_p,1^m;d'_0,\ldots,d'_p)\right) - d_0\cdots d_p d'_0\cdots d'_p\right|$$

$$\leq 10(p+1)k^2(k+1)^2\left(\frac{p}{2^n}+(p+1)\left(\frac{1}{2}\right)^{p+2}\left(\frac{1}{2}\right)^{n+m}\right),$$

hence

$$(28-\text{bis}) \qquad \left|\lambda\left(E_n(d_0,\ldots,d_p,1^m;d'_0,\ldots,d'_p)\right) - d_0\cdots d_p d'_0\cdots d'_p\right|$$

$$\leq 20(p+1)^2k^2(k+1)^2\left(\frac{1}{2}\right)^n.$$

Now formula (α) of Theorem 4.2 is given in $(28)-$bis above, and (β) comes from (28) in the case $p=0$. This ends the proof of Theorem 4.2. ■

Theorem 4.3. *For almost all x, the sequence $(t_n(x))_{n\geq 0}$ is completely uniformly distributed in $[0,1]$, e.g for almost all $x\in[0,1[$ and every $p\geq 0$, the sequence $(t_n(x),\ldots,t_{n+p}(x))_{n\geq 0}$ is uniformly distributed in $[0,1]^{p+1}$. More precisely, for all $\varepsilon>0$ and all $(d_0,d_1,\ldots,d_p)\in[0,1]^{p+1}$, one has*

$$\frac{1}{N}\sum_{n<N}\mathbf{1}_{[0,d_0[\times\cdots\times[0,d_p[}\big(t_n(x),\ldots,t_{n+p}(x)\big) = d_0d_1\ldots d_p + \mathcal{O}\left(\frac{(\log N)^{\frac{3}{2}+\varepsilon}}{\sqrt{N}}\right), \quad \lambda-\text{a.e.}$$

Proof of Theorem 4.3. It is a direct application of Theorem 4.2, (α) and Theorem 11.3 from [Sch]. Indeed, given $p\geq 0$ and $(d_0,\ldots,d_p)\in[0,1]^{p+1}$ from (α), one has, if we let $E_n := E_n(d_0,\ldots,d_p)$,

$$\lambda(E_n) = d_0\ldots d_p + \mathcal{O}(\frac{1}{2^n}),$$

,where the constant in the $\mathcal{O}$ is bounded when $(d_0,\ldots,d_p)$ is fixed, and $E_n(d_0,\ldots,d_p)\cap E_{n+m+p+1}(d_0,\ldots,d_p) = E_n(d_0,\ldots,d_p,1^m,d_0,\ldots,d_p)$, for m large enough. Thus, we can find a convergent series of non negative numbers $(\gamma_k)_{k\geq 0}$ such that $\gamma_k=\mathcal{O}'(\frac{1}{2^k})$, and for any $n\geq 0$ and $t\geq 0$,

$$\lambda(E_n\cap E_{n+t}) \leq \lambda(E_n)\lambda(E_{n+t}) + (\lambda(E_n)+\lambda(E_{n+t}))\gamma_t + \lambda(E_{n+t})\gamma_n. \qquad ■$$

However, using only (β), we have ;

Corollary 4.1. *For $\lambda-$ a.e $x \in [0,1[$, the sequence $\left(t_n(x)\right)_{n\geq 0}$ is uniformly distributed in $[0,1]$ and for all $\varepsilon > 0$, $d \in [0,1]$, and $N \in \mathbf{N}^*$,*

$$A(N,x,d) := \#\{0 \leq n < N;\ 0 \leq t_n(x) < d\} = N.d + \mathcal{O}\left(\sqrt{N}\left(\log(N)\right)^{\frac{3}{2}+\varepsilon}\right).$$

Proof. A straigthforward computation gives

$$\int_0^1 \left|\sum_{n=M+1}^{M+N} \left(\mathbf{1}_{[0,d[}(t_n(x)) - d\right)\right|^2 \lambda(dx) = \mathcal{O}(N),$$

and the corollary results from [Ga-Ko]. ■

Remark 4.1. In a forthcoming paper with A. Thomas, we shall give, as an application, an alternative proof of this fact ([La-Th]). However, the present proof has the advantage that it presents materials that can be quite directly used for proving the non independence, or stochasticity, of the sequence $\left(t_n(\cdot)\right)_{n\geq 0}$.

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