Hitting and return times in ergodic dynamical systems

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Abstract

Given an ergodic dynamical system (X, T, μ), and $U \subset X$ measurable with $\mu(U) > 0$, let $\mu(U)\tau_U(x)$ denote the normalized hitting time of $x \in X$ to U. We prove that given a sequence (U_n) with $\mu(U_n) \to 0$, the distribution function of the normalized hitting time to U_n converges weakly to some pseudo-distribution F if and only if the distribution function of the normalized return time converges weakly to some distribution function $\tilde{F}$, and that in the converging case,

$$F(t) = \int_0^t (1 - \tilde{F}(s))ds,\ t \geq 0. \qquad (\diamondsuit)$$

This in particular characterizes asymptotics for hitting times, and shows that the asymptotic for return times is exponential if and only if the one for hitting times is too.

1. Introduction

In the recent years there has been an interest in the statistics of entry and return times. Typically a neighourhood of a point is considered which can be either a metric ball or a 'cylinder set' associated with a measurable partition. In accordance with a theorem due to Kac one then looks at the return times which are normalised by the measure of the return set. A number of recent papers (e.g. [A1], [A2], [AG], [BV1], [BV2], [C], [CG1], [CO], [GS], [H], [H1], [H2], [HSV], [HV], [P], [PI]) have provided conditions under which this distribution converges to the exponential distribution if the set is shrunk so that its measure converges to zero. On a different note, Lacroix and Kupsa have shown that with a suitable choice of return set one can realise any arbitrarily chosen limiting return time distribution [L] and entry time distribution [K-L] (see Theorem 1).

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The purpose of this note is to show that limiting distributions for entry and return times are intimately linked by the transformation ($\diamondsuit$).

Let $(X, \mathcal{B}, \mu)$ be a probability space, $T\colon X \to X$ a measurable transformation that preserves μ, i.e. $T^*\mu = \mu$. We also assume the dynamical system (X, T, μ) to be ergodic.

For (measurable) $U \subset X$ with $\mu(U) > 0$ we define the **return/entry time function** τ_U by

$$\tau_U(x) = \inf\{k \geq 1 \colon T^k x \in U\}.$$

For $x \in U$, $\tau_U(x)$ denotes the **return time**. On the other hand if we refer to $\tau_U(x)$ as a function on all of X then we call it the **entry time function**. Poincaré's recurrence theorem [K, Theorem 1'] then asserts that τ_U is μ-a.s. well defined. We also have Kac's theorem [K, Theorem 2'] according to which

$$\int_U \tau_U(x)\, d\mu(x) = \sum_{k=1}^{\infty} k\mu(U \cap \{\tau_U = k\}) = 1.$$

Finer statistical properties of the variable $\mu(U)\tau_U$ have been investigated, in a rather large number of recent papers, where particular attention was given to the study of weak convergence of $\mu(U_n)\tau_{U_n}$ as $\mu(U_n) \to 0$. See [AG] for a recent survey in the mixing case.

We say a sequence of distribution functions F_n, $n = 1, 2, \ldots$, **converges weakly** to a function F (which might not be a distribution itselft) if F is increasing and satisfies $\lim_{n\to\infty} F_n(t) = F(t)$ at every point t of continuity of F. (Notice that we require F to be increasing.) We will write $F_n \Rightarrow F$ if F_n converges weakly to F.

Given a $U \subset X$ measurable with $\mu(U) > 0$, we define

$$\begin{gathered}\tilde{F}_U(t) = \tfrac{1}{\mu(U)}\mu\left(U \cap \{\tau_U \mu(U) \leq t\}\right)\\ F_U(t) = \mu(\{\mu(U)\tau_U \leq t\}).\end{gathered}$$

Define

$$\begin{cases} \mathcal{F} = \{F : \mathbb{R} \to [0,1],\ F \equiv 0 \text{ on }]-\infty, 0],\ F \text{ increasing, continuous,} \\ \qquad\qquad\qquad\qquad\qquad \text{concave on } [0, +\infty[,\ F(t) \leq t \text{ for } t \geq 0\}\,; \\ \tilde{\mathcal{F}} = \left\{\tilde{F} : \mathbb{R} \to [0,1],\ \tilde{F} \text{ increasing, } \tilde{F} \equiv 0 \text{ on }]-\infty, 0],\ \int_0^{+\infty}(1 - \tilde{F}(s))ds \leq 1\right\}. \end{cases}$$

These functional classes appear in the following (U_n is always assumed to be measurable):

Theorem 1. *Let (X, T, μ) be an ergodic and aperiodic dynamical system. Then:*

(a) [L] *for any $\tilde{F} \in \tilde{\mathcal{F}}$ there exists a sequence $\{U_n \subset X : n = 1, 2, \dots\}$ such that $\mu(U_n) \to 0$ and $\tilde{F}_{U_n} \Rightarrow \tilde{F}$.*

(b) [K-L] *for any $F \in \mathcal{F}$, there exists $\{U_n \subset X : n = 1, 2, \dots\}$ such that $\mu(U_n) \to 0$ such that $F_{U_n} \Rightarrow F$.*

In this note we prove the following rather unexpected and surprisingly unknown result:

Main Theorem. *Let (X, T, μ) be ergodic, and $\{U_n \subset X : n \geq 1\}$ a sequence of positive measure measurable subsets. Then the sequence of functions $\tilde{F}_{U_n}$ converges weakly if and only if the functions F_{U_n} converge weakly.*

Moreover, if the convergence holds, then

$$F(t) = \int_0^t (1 - \tilde{F}(s))ds, \ t \geq 0, \tag{$\diamondsuit$}$$

where $\tilde{F}$ and F are the corresponding limiting distributions.

The only previous result in this direction was obtained in [HSV] where it is shown that $\tilde{F}_{U_n} \to \tilde{F}$ and $\tilde{F}(t) = 1 - e^{-t}$ for $t \geq 0$ if and only if $F_{U_n} - \tilde{F}_{U_n} \to 0$. Indeed the Main Theorem shows that the exponential distribution if the only fixed point under the transformation ($\diamondsuit$). We state a Corollary:

Corollary 2. *(i) The asymptotic distribution for hitting times, if it exists, is positive exponential with parameter* 1 *if and only if the one for return times is too.*

(ii) Parts (a) and (b) of Theorem 1 are equivalent.

2. Proof of the Main Theorem

For a (measurable) set $U \subset X$ denote by $G_U(T) = \mu\left(\{x \in X : \tau_U(x) \leq T\}\right)$ and similarly $\tilde{G}_U(T) = \mu\left(\{x \in U : \tau_U(x) \leq T\}\right)$. If we denote $V_k = \{x \in U : \tau_U = k\}$ then up to a zero measure set X is the disjoint union of the sets $\bigcup_{j=0}^{k-1} T^j V_k$, $k = 0, 1, \dots$. This in particular implies that

$$G_U(T) = \sum_{j=0}^{k} j\mu(V_j) + \sum_{j=k+1}^{\infty} k\mu(V_j),$$

where $k = [T]$ (integer part). Since the function $G_U(T)$ is constant on intervals that don't contain integers, we get that $G_U'(T) = 0$ if $T \notin \mathbb{N}$ and for $k \in \mathbb{N}$ one has

$$G_U'(k) = \delta_k(G_U(k) - G_U(k-1)) = \delta_k \sum_{j=k}^{\infty} \mu(V_j),$$

where δ_k is the Dirac unit pointmass at k. Since $\tilde{G}_U(T) = \sum_{j=0}^{[T]} \mu(V_j)$ we thus obtain that

$$G'_U(T) = \sum_{k=0}^{\infty} \delta_k(T) \left(\mu(U) - \tilde{G}_U(T) \right)$$

Since $F_U(t) = \frac{1}{\mu(U)} G_U(t/\mu(U))$ and $\tilde{F}_U(t) = \frac{1}{\mu(U)} \tilde{G}_U(t/\mu(U))$, we thus obtain that

$$F'_U(t) = \mu(U) \sum_{k=0}^{\infty} \delta_{k\mu(U)}(t) \left(1 - \tilde{F}_U(t) \right).$$

If we denote by $\bar{F}_U(t)$ the smallest piecewise linear function which is continuous, concave on $[0, +\infty[$ and greater or equal than F_U, then

$$(\star) \qquad \bar{F}'^{+}_U(t) = 1 - \tilde{F}_U(t), \quad t \geq 0,$$

where $\bar{F}'^{+}_U$ denotes the right-hand side derivative. Notice also that

$$(\star\star) \qquad \| F_U - \bar{F}_U \|_\infty \leq \mu(U)$$

since F_U has its discontinuities located at points $\mu(U)$, $2\mu(U)$, $\ldots$.

We continue with the proof of the Main Theorem:

(I) Let us assume there is a sequence of subsets $U_n \subset X$ so that $\mu(U_n) \to 0$ and $\tilde{F}_{U_n} \Rightarrow \tilde{F}$ where $\tilde{F} \in \tilde{\mathcal{F}}$. Since $\tilde{F}$ is increasing, this implies that $\tilde{F}_{U_n} \to \tilde{F}$ Lebesgue almost surely on $[0, +\infty[$. Whence, for given $t \geq 0$, by the Lebesgue dominated convergence theorem on $[0, t]$ ($\tilde{F} \in [0, 1]$), combining with $(\star)$ one has

$$\bar{F}_{U_n}(t) = \int_0^t (1 - \tilde{F}_{U_n}(s)) ds \to \int_0^t (1 - \tilde{F}(s)) ds =: F(t).$$

We put $F(t) = 0$ for $t < 0$. Since $\tilde{F} \in \tilde{\mathcal{F}}$, it follows that $F \in \mathcal{F}$.

Moreover, by $(\star\star)$, $F_{U_n}(t) \to F(t)$ for all $t \in \mathbb{R}$ (the convergence is in fact uniform on compact subsets of $\mathbb{R}$ by [R, Theorem 10.8]).

Hence if $\tilde{F}_{U_n} \Rightarrow \tilde{F}$, then (F_{U_n}) converges weakly to the F associated to $\tilde{F}$ by formula $(\diamondsuit)$.

Proving the reciprocal for the Main Theorem, we need the following:

Lemma 3. *Let f_n, $n = 1, 2, \ldots$, be a sequence of concave functions defined on a non-empty open interval I and assume that f_n converges pointwise to a limit function f. Then off an at most countable subset of I the sequence of derivatives f'_n converges pointwise to the derivative f' of f.*

Proof of Lemma 3. By [R, Theorem 25.3], off an at most countable subset of I, the functions f_n, and f, are differentiable, as concave functions.

Next, using the argument for the proof of [R, Theorem 25.7], but for a fixed $x \in I$ rather than along a sequence of point x_i or points x_i in a closed bounded subset of I, the convergence of the derivatives, when all defined, follows at once. $\square$

(II) Let us now assume that $F_{U_n} \Rightarrow F$. Then [KL] implies that $F \in \mathcal{F}$. Whence by ($\star$) and ($\star\star$), we have, for $t \geq 0$,

$$\bar{F}_{U_n}(t) = \int_0^t \bar{F}'_{U_n}{}^+(s)ds = \int_0^t (1 - \tilde{F}_{U_n}(s))ds \to F(t) \ (= \int_0^t F'^+(s)ds).$$

It now follows from Lemma 3 that off an at most countable subset Ω of $]0, +\infty[$, the functions $1 - \tilde{F}_{U_n}(s)$ converge pointwise to $F'^+(s)$. Put $\tilde{F}(s) := 1 - F'^+(s)$ for $s \in \mathbb{R}$.

It remains to show that $\tilde{F}_{U_n}(s) \to \tilde{F}(s)$ at points s of continuity of F'^+. Clearly if $s \notin \Omega$ or $s < 0$ there is nothing to do. Else, for any $s_1 < s < s_2$ not in Ω, we have

$$\tilde{F}(s_1) \leq \liminf_n \tilde{F}_{U_n}(s) \leq \limsup_n \tilde{F}_{U_n}(s) \leq \tilde{F}(s_2),$$

and since Ω is dense in $[0, +\infty[$, the conclusion follows. So $\tilde{F}_{U_n} \Rightarrow \tilde{F} = 1 - F'^+$, which ends the proof. $\square$

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