MERIT FACTORS AND MORSE SEQUENCES

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Abstract. We show that Turyn's conjecture, arising from the Theory of Error Correcting Codes, has an equivalent formulation in Dynamical Systems Theory. In particular, Turyn's conjecture is true if all binary Morse flows have singular spectra. Our proof uses intermediate estimates for merit factors of products of words, and is purely combinatorial.

Résumé. Nous montrons que la conjecture de Turyn, issue de la Théorie des Codes Correcteurs d'Erreur, a une formulation équivalente en Théorie des Systèmes Dynamiques. En particulier, la première est vraie si tous les flots de Morse continus binaires ont un spectre singulier. La preuve utilise des estimations intermédiaires du facteur de mérite d'un produit de mots, et repose sur des méthodes purement combinatoires.

Introduction

In the Theory of Error Correcting Codes [M-S], the basic object to deal with is a *binary code*, which consists of a finite family of ± 1-valued finite strings (or code words). Their use in data transmission and radar systems require, for optimal efficiency and/or minimal source detection, that certain positive real valued quantities, the *merit factors* of the code words - to be defined below, be maximal.

These quantities are closely connected to L^4-norms of trigonometric polynomials associated to code words (see Lemma 0 in this introduction), and at once exhibit connections with so-called "ultraflat" problems, well-known in Harmonic Analysis. Such problems (with L^4-norm replaced by L^2-norm) have recently been shown to have equivalent formulations in Dynamical Systems Theory [B], [Gu], which are connected to a weak form of a still open question of S. Banach.

In this note we shall show that the same holds for L^4-norm (Theorem 2). However, we use completely different techniques, which are purely combinatorial, and require very little background on Dynamical Systems Theory. Some of our intermediate computations (Theorem 1) give estimates for merit factors of products of words.

Before entering the subject, we would like to point out to the reader's attention that several normed spaces shall be used hereafter : let us, for clarity, indicate them :

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– the space $\mathbb{C}$ of complex numbers z, with conjugate $\bar{z}$, and modulus $|z|$;

– the spaces L^p $(p \geq 1)$ of the multiplicative group $\{|z| = 1\}$ equipped with Lebesgue one-dimensional measure (its normalized Haar measure), used essentially for trigonometric polynomials;

– the spaces $\ell^p = \ell^p(\mathbb{N})$ of complex valued sequences, with usual ℓ^p-norm;

– the spaces $L^p(\mu)$, where μ is a Borel probability measure on some abstract compact metric space X.

Let us now introduce the basic definitions :

Definition 1. Let $A = A(0)A(1)...A(a-1) \in \mathbb{C}^a$, $a \in \mathbb{N}$ (such A will be called a *word*). The *aperiodic autocorrelation function* of A is defined on $\{0, 1, ..., a-1\}$ by

$$\Phi_A(n) = \frac{1}{a} \sum_{k=0}^{a-1-n} A(k)\overline{A(k+n)}.$$

The word A is *normalized* if $\Phi_A(0) = 1$ (each binary word is automatically normalized).

The values of Φ_A give rise to the following parameter:

Definition 2. The *merit factor* of A is $\frac{1}{2M_A}$, where

$$M_A = \sum_{n=1}^{a-1} |\Phi_A(n)|^2.$$

If A is a normalized word, we define the trigonometric polynomial

$$P_A(z) = \frac{1}{\sqrt{a}} \sum_{n=0}^{a-1} A(n) z^n.$$

Lemma 0. *If A is a normalized word then $\|P_A\|_2 = 1$ and $2M_A = \|P_A\|_4^4 - 1$.*

Proof. Write $P_A(z) = \frac{1}{\sqrt{a}} \sum_{n=0}^{a-1} A(n) z^n$. Then

$$|P_A(z)|^2 = \frac{1}{a} \sum_{n=-(a-1)}^{a-1} \Big(\sum_{\substack{0 \leq s,t < a, \\ s-t=n}} A(s)\overline{A(t)} \Big) z^n = \sum_{n=-(a-1)}^{a-1} B(n) z^n,$$

where $B(n) = \frac{1}{a} \sum_{\substack{0 \leq s,t < a, \\ s-t=n}} A(s)\overline{A(t)}$ equals $\Phi_A(n)$ if $n \geq 0$, and $\overline{\Phi_A(n)}$ otherwise. Hence $\|P_A\|_2 = \sqrt{B(0)} = \sqrt{\Phi_A(0)} = 1$ if A is normalized.

Next, $|P_A(z)|^4 = \sum_{n=-2(a-1)}^{2(a-1)} \big(\sum_{\substack{0 \leq |s|,|t| < a, \\ s+t=n}} B(s)\overline{B(t)} \big) z^n$, thus

$$\begin{aligned} \|P_A\|_4^4 &= \sum_{\substack{0 \leq |s|,|t| < a, \\ s+t=0}} B(s)\overline{B(t)} \\ &= |B(0)|^2 + 2 \sum_{s=1}^{a-1} B(s)\overline{B(s)} \\ &= 1 + 2M_A. \end{aligned}$$

□

The above mentioned interest in applications is to produce code words having as big as possible merit factor, or, equivalently, by Lemma 0, trigonometric polynomials with L^2-norm equal to 1 and L^4-norm as close as possible to 1.

Highest merit factors within binary words are obtained for so-called *Barker sequences*, for which the aperiodic autocorrelation function assumes lowest possible values, i.e., 0 for even arguments, and $\pm 1/a$ for the odd ones, which leads to largest possible merit factor $a^2/(a-1)$ for that length a of a binary word. Unfortunately, the longest Barker sequence identified is

$$1 \ -1\ 1\ \ -1\ 1\ 1\ \ -1\ \ -1\ 1\ 1\ 1\ 1\ 1.$$

What is "even worse", the above Barker sequence has the largest merit factor 14.08 ever known [G]. In [G], it was conjectured that this number can not be improved. Using theoretical methods aided by fast computer programs Barker sequences of lengths between 14 and 1898884 have been proved not to exist [E-K]. The possible lengths of Barker sequence have also been restricted seriously [E-K].

Nobody has proved yet that the merit factors of binary words are bounded (the hypothesis on boundedness is called the *Turyn's conjecture*, or, equivalently, the *Erdös L_4-norm conjecture* [S-S]), neither that there exist only finitely many Barker sequences.

We refer the reader to [C-S], [H-J-J], and [M-S] for more information concerning the merit factor problem.

The merit factor of a product of words

This section contains the computational results of this note.

Let $A = A(0)\dots A(a-1)$ and $B = B(0)\dots B(b-1)$ be two words. We define their *product* as the word $A \times B$, of length ab, by

$$A \times B(s+at) = A(s)B(t), \quad 0 \le t < b,\ 0 \le s < a.$$

Lemma 1. *Let $A = A(0)\dots A(a-1)$ and $B = B(0)\dots B(b-1)$ be two normalized words. Let $0 \le n \le ab-1$, $n = s+at$, $0 \le s \le a-1$ and $0 \le t \le b-1$. Then*

$$\Phi_{A\times B}(n) = \Phi_A(s)\Phi_B(t) + \overline{\Phi_A(a-s)}\Phi_B(t+1)$$

(with the convention that $\Phi_A(a) = 0$). In particular, $A \times B$ is normalized.

Proof. Let $0 \le k \le ab-1-n$. We have $k = i + aj$ for some $0 \le i \le a-1$ and $0 \le j \le b-1$, where either $i+s \le a-1$ (then $j+t \le b-1$) or $i+s \ge a$ (then $j+t \le b-2$). For fixed n we denote the sets of k's of the first and second above cases by K_0 and K_1, respectively. We have

$$\Phi_{A\times B}(n) = \frac{1}{ab}\sum_{k=0}^{ab-1-n}(A\times B)(k)\overline{(A\times B)(k+n)} =$$

$$\frac{1}{ab}\sum_{k\in K_0} A(i)B(j)\overline{A(i+s)B(j+t)} + \frac{1}{ab}\sum_{k\in K_1} A(i)B(j)\overline{A(i+s-a)B(j+t+1)} =$$

$$\frac{1}{ab}\sum_{i=0}^{a-1-s}\sum_{j=0}^{b-1-t} A(i)\overline{A(i+s)}B(j)\overline{B(j+t)}+$$

$$\frac{1}{ab}\sum_{i=a-s}^{a-1}\sum_{j=0}^{b-2-t} A(i)\overline{A(i+s-a)}B(j)\overline{B(j+t+1)} =$$

$$\frac{1}{a}\sum_{i=0}^{a-1-s} A(i)\overline{A(i+s)}\ \frac{1}{b}\sum_{j=0}^{b-1-t} B(j)\overline{B(j+t)}+$$

$$\frac{1}{a}\sum_{i'=0}^{a-1-s'} \overline{A(i')}A(i'+s')\ \frac{1}{b}\sum_{j=0}^{b-1-t'} B(j)\overline{B(j+t')} =$$

$$\Phi_A(s)\Phi_B(t) + \overline{\Phi_A(s')}\Phi_B(t'),$$

by change of indices $s' = a-s$, $i' = i+s-a$, $t' = t+1$. □

Theorem 1. *The following inequalities hold for normalized words:*

$$M_A + M_B + 2M_AM_B - 2M_A\sqrt{M_B^2 + M_B}$$
$$\le M_{A\times B} \le$$
$$M_A + M_B + 2M_AM_B + 2M_A\sqrt{M_B^2 + M_B}.$$

Proof. Applying Lemma 1, we obtain

$$M_{A\times B} = \sum_{n=1}^{ab-1}|\Phi_{A\times B}(n)|^2 = \sum_{s=1}^{a-1}\sum_{t=1}^{b-1}\left|\Phi_A(s)\Phi_B(t) + \overline{\Phi_A(a-s)}\Phi_B(t+1)\right|^2 +$$
$$\sum_{t=1}^{b-1}|\Phi_B(t)|^2 + \sum_{s=1}^{a-1}\left|\Phi_A(s) + \overline{\Phi_A(a-s)}\Phi_B(1)\right|^2, \quad (1)$$

the last two single sums representing the terms of the double sum for $s=0$ and for $t=0$, respectively. Developing squared sums and applying the equality $\Phi_B(b) = 0$

where necessary, we obtain:

$$M_{A\times B} = \sum_{s=1}^{a-1} |\Phi_A(s)|^2 \sum_{t=1}^{b-1} |\Phi_B(t)|^2 + \sum_{s=1}^{a-1} |\overline{\Phi_A(a-s)}|^2 \sum_{t=1}^{b-2} |\Phi_B(t+1)|^2 +$$
$$2\mathrm{Re} \sum_{s=1}^{a-1} \Phi_A(s)\Phi_A(a-s) \sum_{t=1}^{b-1} \Phi_B(t)\overline{\Phi_B(t+1)} +$$
$$\sum_{t=1}^{b-1} |\Phi_B(t)|^2 + \sum_{s=1}^{a-1} |\Phi_A(s)|^2 + \sum_{s=1}^{a-1} |\overline{\Phi_A(a-s)}|^2 |\Phi_B(1)|^2 +$$
$$2\mathrm{Re} \sum_{s=1}^{a-1} \Phi_A(s)\Phi_A(a-s)\ \overline{\Phi_B(1)} = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6 + \Sigma_7 \quad (2)$$

Clearly, $\Sigma_1 = \Sigma_2 + \Sigma_6 = M_A M_B$, $\Sigma_4 = M_B$, $\Sigma_5 = M_A$. Further, estimating the inner product of two vectors by the product of their lengths, we obtain

$$\left| \sum_{s=1}^{a-1} \Phi_A(s)\Phi_A(a-s) \right| \leq M_A \quad \text{and} \quad \left| \sum_{t=0}^{b-1} \Phi_B(t)\overline{\Phi_B(t+1)} \right| \leq \sqrt{M_B+1}\sqrt{M_B},$$

hence, using $\Phi_B(0) = 1$, we have $|\Sigma_3 + \Sigma_7| \leq 2M_A\sqrt{M_B^2 + M_B}$ and the assertion is proved. □

Corollary 1. *The sums* $\Sigma_5,\ \Sigma_6,\ \Sigma_7$ *defined in the preceding proof satisfy*

$$\Sigma_5 + \Sigma_6 + \Sigma_7 \geq M_A(1 - |\Phi_B(1)|)^2.$$

Proof. $\Sigma_6 = M_A|\Phi_B(1)|^2$, $\Sigma_7 \geq -2M_A|\Phi_B(1)|$. □

Corollary 2. *Another possible lower estimate for* $M_{A\times B}$ *is*

$$M_B + M_A(1 - 2|\Phi_B(1)|).$$

Proof. Use $\Sigma_3 \geq -2M_A M_B$. □

Dynamical Systems

Let us assume that we are given a triple (X, T, μ) where X is a compact metric space, $T : X \to X$ is a homeomorphism, and μ is a Borel probability measure on X such that $T\mu = \mu$ (the existence of at least one such T-invariant measure is well known).

Then we consider a unitary operator $U_T : L^2(\mu) \to L^2(\mu)$ defined by $U_T(f) = f \circ T$. The measure μ is called *ergodic* if all U_T-invariant functions in $L^2(\mu)$ are constants. Ergodic measures are characterized as the extreme points of the (convex) set of all T-invariant measures.

The dynamical system (X, T) is called *uniquely ergodic* if there exists only one T-invariant measure μ. Clearly, in this case μ is ergodic.

Spectral preliminaries.

We now provide the minimum background in Spectral Theory of Unitary Operators, necessary for the purposes of this note.

Consider a unitary operator U on a separable Hilbert space H (in our case this will be the operator U_T on $L^2(\mu)$). Due to the classical Bochner-Hergoltz theorem, for each element (function) $f \in H$, we can identify its *spectral measure* μ_f on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The measure μ_f is determined by the property that for every $n \in \mathbb{N}$

$$\int_{\mathbb{T}} z^n d\mu_f = \int_X f\overline{U^n(f)}d\mu.$$

The above numbers are denoted by $\hat{\mu}_f(n)$ and called the *Fourier coefficients* of μ_f.

Another classical theorem (based on a theorem of Wiener) assures that there exists a decomposition

$$H = \bigoplus_{i \in E} H_i,$$

where E is an interval of integers $[0, m)$ or $[0, \infty)$, and H_i is the closed linear invariant subspace of H generated by a single element f_i

$$H_i = \overline{\mathrm{lin}\{U^n(f_i) : n \in \mathbb{Z}\}}.$$

Moreover, we can organize the above decomposition so that for each $i \in E, i > 0$, μ_{f_i} is absolutely continuous with respect to $\mu_{f_{i-1}}$. The cardinality m (or ∞) of E is called the *spectral multiplicity* of U. The above sequence of spectral measures $(\mu_{f_i})_{i \in E}$, more precisely their types (by the *type* of a measure one understands the equivalence class with respect to absolute continuity) provide a complete spectral isomorphism invariant, i.e., if two unitary operators have the same spectral multiplicity and each of the above spectral measures for one of them is equivalent to the corresponding measure of the other then the operators are conjugate via an isometry. This explains why spectral multiplicity and spectral types are so important.

We say that U has *simple spectrum* if $m = 1$. The operator U has *Lebesgue spectrum* if μ_{f_0} is equivalent to the Lebesgue measure λ on the circle. Similarly, we say that U has *singular spectrum* if μ_{f_0} (and hence μ_{f_i} for all $i \in E$) is singular with respect to λ. For example, if the sequence $(\hat{\mu}_f(n))_{n \in \mathbb{N}}$ is square summable (which we denote by $\hat{\mu}_f \in \ell^2$) then the measure μ_f is absolutely continuous with respect to λ and has density

$$\frac{d\mu_f}{d\lambda} = \hat{\mu}_f(0) + \sum_{n \in \mathbb{N}} 2\mathrm{Re}(\mu_f(n) z^n),$$

which belongs to $L^2(\lambda)$.

The same spectral properties are defined for dynamical systems (X, T, μ) by referring to the induced unitary operator U_T on $L^2(\mu)$. The case of a uniquely ergodic dynamical system is the most convenient here, because then there is no need in specifying which invariant measure is being considered.

We refer the interested reader to [P] for further information concerning Dynamical Systems Theory.

Morse binary flows.

Again, only minimum necessary information will be provided here. We refer for further information to [K1] or [Gu].

Definition 3. A sequence of binary words $B_1, B_2, ...$ satisfying $B_p(0) = 1$ for any $p \in \mathbb{N}$, determines a (one-sided, generalized) *Morse sequence* A as the coordinate-wise limit of the words $A_p = B_1 \times B_2 \times \cdots \times B_p$ (convergence is granted by the condition $B_p(0) = 1$).

Every such Morse sequence A can be easily extended to a bi-infinite sequence A' such that every word which appears in A' also appears in its positive part A. For example, at each stage we can extend A_p by attaching on it's left another copy of A_p. The obtained words A_pA_p extend over the coordinates $[-a_p, a_p)$ (with zero coordinate approximately in its center). In fact, they may not converge, but any convergent subsequence provides a desired bi-infinite extension; it is not hard to see that each word A_pA_p appears further in the positive part A (the last statement fails for some periodic Morse sequences, but these are easy to extend by periodicity).

Next we consider the shift transformation S on the set of bi-infinite sequences given by $Sx(n) = x(n+1)$ $(n \in \mathbb{Z})$, and we define a *Morse flow* as the dynamical system (X, S), where $X = \overline{\{S^nA' : n \in \mathbb{Z}\}}$ and A' is a bi-infinite extension of a Morse sequence A.

Morse flows have been extensively studied for their dynamical and spectral properties. For us it is important to know the following facts:

Fact 1. (see [I-L] and the reference therein) *A sufficient condition for a Morse flow to be uniquely ergodic is that the frequencies of both letters* -1 *and* 1 *in the words* B_p *are bounded away from zero.*

Fact 2. (see e.g. [Kw]) *A uniquely ergodic binary Morse flow has simple spectrum.*

(Unique ergodicity is not an essential assumption here. There always exists an invariant measure with respect to which the system has simple spectrum. Our assumption only assures that we are not looking at a wrong invariant measure.)

Weak form of the Banach's question.

Not all possible spectra of unitary operators can be realized by unitary operators arising from dynamical systems. One of the central problems in ergodic theory is to describe the admissible spectra. A still unsolved question is that due to S. Banach, addressing to the existence of simple Lebesgue spectrum for some (X, T, μ).

Does there exist an ergodic (X, T, μ) with simple spectrum and a Lebesgue component? This question is still open in Dynamical Systems Theory, also. Even if simple pure Lebesgue spectrum was not admitted for transformations (i.e., if the answer to the Banach's question was negative) one might still try to construct a dynamical system with a simple spectrum consisting e.g. of the Lebesgue measure plus some atoms.

The answer to this weak form of the Banach's question is unknown except for some very special recently examined cases leading to certain generalized Riesz products proved to be almost surely singular (with respect to an appropriate natural probability on families of dynamical systems, see [C-N], [D-E], [Gu]).

There is still hope to obtain a positive answer by an effective construction. Various Morse flows exhibiting coexistence of numerous measure theoretic and spectral phenomena have been constructed in recent years (e.g. [K2], [L1], [L2], [K-L]). Thus it seems natural to search for solution of this problem within the class of Morse flows.

There have been attempts made toward constructing an appropriate binary Morse flow (X, S, μ) such that the "zero coordinate function" $f(x) = x(0)$ $(x \in X)$

has absolutely continuous spectral measure, which would be verified by the condition $\hat{\mu}_f \in \ell^2$. The above condition is even stronger than we need - it says that μ_f is absolutely continuous and has density in $L^2(\lambda)$, while having density in $L^1(\lambda)$ would be sufficient. But there is no convenient criterion allowing to verify the L^1 property by looking only at the Fourier coefficients. In what follows we will be using the L^2 condition, and we prove that the success of such an attempt fully depends on the solution of the merit factor problem. (In the introduction we have mentioned the L^4-norm of certain polynomials. There is no mistake. Roughly speaking, the density of the spectral measure of the obtained Morse flow corresponds to the limit of squares of the polynomials $P_{A_p}(z)$, hence its L^2-norm exists if and only if the L^4-norms of the polynomials converge).

Turyn's conjecture and spectral properties of Morse flows

First we need to define the merit factor for one-sided infinite sequences.

The autocorrelation function of an infinite sequence $A = A(0)A(1)A(2)... \in \mathbb{C}^{\mathbb{N}}$ is defined for each $n \in \mathbb{N}$ as the limit $\Phi_A(n) = \lim_{a\to\infty} \Phi_{A_a}(n)$, where A_a is the finite word $A(0)A(1)...A(a-1)$. Of course, there is no guarantee that such a limit exists. If $\Phi_A(n)$ are well defined for all $n \in \mathbb{N}$, then we let

$$M_A = \sum_{n=1}^{\infty} |\Phi_A(n)|^2.$$

Lemma 2. *If a Morse sequence A is obtained from normalized words $B_1, B_2, ...$ as in Definition 3, and if M_A is well defined then $M_A = \lim_p M_{A_p}$.*

Proof. The inequality $M_A \le \lim M_{A_p}$ follows at once from the convergence $\Phi_{A_p}(n) \to \Phi_A(n)$ for each n. The converse inequality is trivial if $M_A = \infty$. The case $M_A < \infty$ is a bit more complicated.

First observe that, by grouping:

$$\begin{aligned} B'_1 &= B_1 \times B_2 \times ... \times B_{p_1}, \\ B'_2 &= B_{p_1+1} \times B_{p_1+2} \times ... \times B_{p_2}, \\ &... \end{aligned}$$

we can assume that the lengths a_p of the words A_p grow sufficiently fast, so that the values of the autocorrelation function for all arguments up to a_p evaluated for A and for A_{p+1} are almost the same. More precisely, we can assume that

$$\sum_{n=1}^{a_p} |\Phi_{A_{p+1}}(n) - \Phi_A(n)|^2 \to 0,$$

which implies in particular that the sums

$$S_p = \sum_{n=1}^{a_p-1} |\Phi_{A_{p+1}}(n)|^2$$

converge to M_A.

We shall show that $\Phi_{B_p}(1) \to 0$. Indeed, if not, then $|\Phi_{B_p}(1)| \geq \delta$ for some $\delta > 0$ and infinitely many indices p. But we have, by Lemma 1, $\Phi_{A_{p+1}}(a_p) = \Phi_{B_{p+1}}(1)$ (here $t = 1$ and $s = 0$), which implies that $|\Phi_A(a_p)|$ is greater than $\delta/2$ for infinitely many indices p, thus $M_A = \infty$, a contradiction.

To conclude, recall that $A_{p+1} = A_p \times B_{p+1}$ and observe that S_p is the part corresponding to $t = 0$ in the sum representing $M_{A_p \times B_{p+1}}$, i.e., to the last item of formula (1) which then becomes $\Sigma_5 + \Sigma_6 + \Sigma_7$ in (2). By Corollary 1, this is estimated from below by $M_{A_p}(1 - |\Phi_{B_{p+1}}(1)|)^2$ which converges to the same limit as M_{A_p}. We have proved that $M_A = \lim S_p \geq \lim M_{A_p}$, as desired. □

We are in a position to state our main result.

Theorem 2. *There exist binary words with arbitrarily large merit factors if and only if there exists a binary Morse flow (X, S, μ) such that the Fourier transform of the spectral measure μ_f of the "zero coordinate function" belongs to ℓ^2. In particular, the spectrum of (X, S, μ) is then simple and not purely singular.*

Proof. Let $B_1, B_2, \ldots$ be finite binary sequences such that $M_{B_p} \to 0$. By choosing a subsequence we can assume that the speed of the convergence is sufficient, the precise meaning of which will be specified later. We can also assume $B_p(0) = 1$, since this amounts only to multiplying (if necessary) a B_p by $B_p(0)$, which does not change M_{B_p}. Let A be the Morse sequence obtained from (B_p).

Observe that from the assumption $M_{B_p} \to 0$ it follows automatically that the frequencies of -1 and 1 in B_p both converge to 1/2, hence they are bounded away from zero, which in turn implies that the generated Morse flow is uniquely ergodic (Fact 1).

Thus we have a uniquely ergodic flow (X, S, μ), with simple spectrum (Fact 2). Define f by $f(x) = x(0)$ $(x \in X)$. Clearly $f \in L^2(\mu)$. The Fourier coefficients of the spectral measure μ_f of f are

$$\hat{\mu}_f(n) = \int f\, \overline{U_S^n(f)} d\mu$$

(note that $U_S^n(f)(x) = x(n)$). Since f is continuous, and by unique ergodicity, we can evaluate the integrals by taking averages along the trajectory of the element A:

$$\hat{\mu}_f(n) = \lim_{a \to \infty} \frac{1}{a} \sum_{k=0}^{a-1} A(k)\overline{A(k+n)} = \Phi_A(n),$$

and these limits exist for all n.

In order to have $\hat{\mu}_f \in \ell^2$, it suffices that M_A be finite, because $\sqrt{M_A}$ is the ℓ^2 norm of the sequence $(\Phi_A(n))$. By Lemma 2, it now suffices to have M_{A_p} bounded. Finally, from the upper estimate of Theorem 1 applied to $A_{p+1} = A_p \times B_{p+1}$, it is seen that this goal can be achieved by assuming a sufficient speed of the convergence $M_{B_p} \to 0$, which we can do (this is how we specify the meaning of a "sufficient speed").

Conversely, as shown in the proof of Lemma 2, the convergence $\Phi_{B_p}(1) \to 0$ is necessary for M_A to be finite. Then by Corollary 2, it is seen that if M_{B_p} are bounded away from zero then M_{A_p} diverge, hence $M_A = \infty$. In other words, if merit factors of all binary words are bounded, then there are no chances to construct a binary Morse flow for which $\hat{\mu}_f \in \ell^2$. We have established the announced equivalence. □

Corollary 3. *If all continuous binary Morse flows have singular spectra (in particular if the weak version of Banach's question has negative answer) then the merit factors of binary words are bounded (the Turyn's conjecture holds), in particular there are only finitely many Barker sequences.*

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