NUMBER SYSTEMS AND REPARTITION

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Abstract. Given a number system $\mathcal{S} = \Big((0,1), T_j, (J_{n,j})_{n\in E_j}\Big)_{j\geq 0}$, we define a measurable mapping $\Phi_{\mathcal{S}} : (0,1)^{\mathbb{N}} \to (0,1)$ such that $\lambda_\infty(\Phi_{\mathcal{S}}^{-1}(A)) = \lambda(A)$, $A \in \mathcal{B}_{(0,1)}$. A measurable section $\big(t_n(.)\big)_{n\geq 0}$ is defined for $\Phi_{\mathcal{S}}$; $t_n(.)$ has uniform distribution for any $n \geq 0$. Conditions relative to λ–a.e. repartition properties of $\big(t_n(.)\big)_{n\geq 0}$ are studied. Applications to (α,γ)–expansions, Cantor products and Continued fractions are given.

0. INTRODUCTION.

$\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. In the following, when we consider sub-intervals of the unit interval, (a,b) stands for one of the intervals $]a,b[$, $[a,b[$, $[a,b]$, $]a,b]$.

For $p \in \mathbb{N}^*$, we denote by $\mathcal{B}_{(0,1)^p}$ the Borelian σ-algebra of $(0,1)^p$, and $\lambda_p = \displaystyle\bigotimes_{i=0}^{p-1} \lambda$, where λ is the Lebesgue measure on $(0,1)$. Furthermore, let $\lambda_\infty = \displaystyle\bigotimes_{i\in\mathbb{N}} \lambda$ be the product measure on $\mathcal{B}_{(0,1)^{\mathbb{N}}}$. If $u \in (0,1)^{\mathbb{N}}$, $u = (u_0, \dots, u_n, \dots)$, and $p \in \mathbb{N}^*$, let $u^{(p)} = \big((u_i, \dots, u_{i+p-1})\big)_{i\geq 0}$.

We call a triple $\big((0,1), T, (J_n)_{n\in E}\big)$ a fibered system if $T : (0,1) \to (0,1)$ is measurable, $(J_n)_{n\in E}$, is a partition of $(0,1)$ into intervals with positive length, and $T_{|J_n}$ (restriction) is one to one (cf [11]).

A sequence of fibered systems $\mathcal{S} = \Big((0,1), T_j, \big(J_{n,j}\big)_{n\in E_j}\Big)_{j\geq 0}$ is called a Number System, if the following conditions hold:

$$\begin{cases} (A) & \forall j \geq 0,\ \big((0,1), T_j, (J_{n,j})_{n\in E_j}\big) \text{ is a fibered system, and } T_0 = Id_{(0,1)}; \\ (B) & \begin{cases} \forall j \geq 1, \text{ put } C_j = T_{j-1} \circ \cdots \circ T_0; \text{ let } B(n_1, \dots, n_p) := \\ \{x,\ \forall i \in [1,p],\ C_i(x) \in J_{n_i,i}\}. \text{ Then if } B(n_1,\dots,n_p) \neq \emptyset, \text{ assume} \\ B(n_1,\dots,n_p) = (a_{n_1\dots n_p}, b_{n_1,\dots,n_p}) \text{ and } \lambda(B(n_1,\dots,n_p)) > 0; \end{cases} \\ (C) & \begin{cases} \text{If } (n_i)_{i\geq 1} \in \prod_{i=1}^{+\infty} E_i, \text{ and for any } p \geq 1,\ B(n_1,\dots,n_p) \neq \emptyset, \\ \text{then } \bigcap_{p\geq 1} B(n_1,\dots,n_p) \text{ is a singleton.} \end{cases} \end{cases}$$

If $\mathcal{S}$ is a number system, then for any $x \in (0,1)$, there exists a unique $\big(n_i(x)\big)_{i\geq 1} \in \prod_{i\geq 1} E_i$ such that $\{x\} = \cap_{p\geq 1} B(n_1(x), \dots, n_p(x))$. Conversely, if $(n_i)_{i\geq 1} \in \prod_{i\geq 1} E_i$ is such that $\cap_{p\geq 1} B(n_1,\dots,n_p) \neq \emptyset$, it defines a unique element $x \in (0,1)$ satisfying $(n_i(x))_{i\geq 1} = (n_i)_{i\geq 1}$, given by $\{x\} = \cap_{p\geq 1} B(n_1,\dots,n_p)$. The sequence $(n_i(x))_{i\geq 1}$ is the digital representation of x.

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This definition of a number system is not the most general one would expect, since it could be necessary to delete some subset from $(0,1)$. However, the results that follow still hold if we do so.

As usual, realizable finite sequences in $\prod_{i=1}^{p} E_i$ (or infinite sequences in $\prod_{i=1}^{+\infty} E_i$) are defined by

$$\begin{cases} R_p(\mathcal{S}) = \{(n_1, \dots, n_p) \in \prod_{i=1}^{p} E_i,\ B(n_1, \dots, n_p) \neq \emptyset\}, \\ R(\mathcal{S}) = \{(n_i)_{i \geq 1} \in \prod_{i=1}^{+\infty} E_i,\ \forall p \geq 1,\ B(n_1, \dots, n_p) \neq \emptyset\}. \end{cases} \tag{1}$$

Let $\mathcal{S}$ be a number system: with the notations above, let $\Phi_{\mathcal{S}} : (0,1)^{\mathbb{N}} \to (0,1)$ be defined as follows. For $u = (u_0, \dots, u_n, \dots) \in (0,1)^{\mathbb{N}}$ one first defines the sequence $(n_i)_{i\geq 1}$: choose n_1 such that $u_0 \in B(n_1)$. Then inductively, given $B(n_1, \dots, n_m)$, choose n_{m+1} such that

$$(1-u_m)a_{n_1\dots n_m} + u_m b_{n_1\dots n_m} \in B(n_1, \dots, n_m, n_{m+1}). \tag{2}$$

Condition (C) ensures the existence of a unique $x \in (0,1)$ such that $(n_i(x))_{i\geq 1} = (n_i)_{i\geq 1}$. Finally, put $\Phi_{\mathcal{S}}(u) = x$.

Now define $(t_n : (0,1) \to (0,1))_{n\geq 0}$ by

$$\begin{cases} t_0(x) = x = \frac{x-a_\emptyset}{b_\emptyset - a_\emptyset} \text{ where } a_\emptyset = 0,\ b_\emptyset = 1; \\ t_m(x) = \frac{x - a_{n_1\dots n_m(x)}}{b_{n_1\dots n_m(x)} - a_{n_1\dots n_m(x)}},\ m \geq 1. \end{cases} \tag{3}$$

One has $\Phi_{\mathcal{S}}\left((t_n(x))_{n\geq 0}\right) = x$. Indeed, with (C), (2) and (3), this follows from

$$\begin{cases} x \in J_{n_1(x),1}, \\ (1-t_m(x)) \cdot a_{n_1(x)\dots n_m(x)} + t_m(x) \cdot b_{n_1(x)\dots n_m(x)} = x \in B(n_1(x)\dots n_{m+1}(x)). \end{cases}$$

Thus, $\big(t_n(.)\big)_{n\geq 0}$ is a measurable section for $\Phi_{\mathcal{S}}$.

Remark 0.1. *Assume $T_j = T$, $j \geq 1$. Then define $\sigma(u) = (u_{n+1})_{n\geq 0}$, the shift transformation on $(0,1)^{\mathbb{N}}$. Obviously, the relation $\Phi_{\mathcal{S}} \circ \sigma = T \circ \Phi_{\mathcal{S}}$ holds λ_∞-a.e. on $(0,1)^{\mathbb{N}}$ if and only if on $J_{n,j} := J_n = (a_n, b_n)$, $T(x) = \frac{x-a_n}{b_n-a_n}$, $n \in E := E_j$. Accordingly, $t_n(.) = T^n$, $n \geq 0$. Note that for such transformations, $\lambda(T^{-1}(B)) = \lambda(B)$, $B \in \mathcal{B}_{(0,1)}$.*

The following statement holds: if $\Phi_{\mathcal{S}}(u) = x$, then for any $m \geq 0$

$$|u_m - t_m(x)| \leq \frac{\lambda\big(B(n_1 \dots n_{m+1}(x))\big)}{\lambda\big(B(n_1 \dots n_m(x))\big)}, \tag{4}$$

where $B(\emptyset) = (a_\emptyset, b_\emptyset) = (0,1)$.

The paper is organized as follows. Part 1 states the main elementary property of $\Phi_{\mathcal{S}}$ (THEOREM 1.1.) and uniform distribution of the random variable $t_n(.)$, $n \geq 0$ (PROPOSITION 1.1.). A criterion for two sequences to be simultaneously completely uniformly distributed (respectively for two such to have same logarithmic p-dimensional discrepancies for all p) is given in LEMMA 1.1. (respectively LEMMA 1.2.). In Remark 1.1. we apply THEOREM 1.1. to recover a previous result of [1].

Part 2 gives a sufficient condition for the sequence $\big(t_n(.)\big)_{n\geq 0}$ to be λ-a.e. completely uniformly distributed (THEOREM 2.1.), and another one for it to have λ-a.e. infinite p-dimensional logarithmic discrepancy, for any p (THEOREM 2.2.). These properties are in fact carried over from the corresponding ones on $(0,1)^{\mathbb{N}}$, using $\Phi_{\mathcal{S}}$.

Part 3 gives applications of parts 1 and 2, in the cases of (α,γ)-expansions (cf [1]), generalized Cantor products (cf [7]), and continued fractions (cf [5]). One can find for Engel or Sylvester series improvements of the previous results concerning uniform distribution of the relevant sequences $\big(t_n(.)\big)_{n\geq 0}$ (cf [11, Chapter 11]). THEOREM 3.1. gives a sufficient condition for the sequence $\big(t_n(.)\big)_{n\geq 0}$ to be λ-a.e. uniformly distributed; this applies to β-expansions (cf [9]) and continued fractions.

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1. PRELIMINARY RESULTS AND DEFINITIONS.

The map $\Phi_{\mathcal{S}}$ satisfies

THEOREM 1.1. *For any $A \in \mathcal{B}_{(0,1)}$, $\lambda_\infty\big(\Phi_{\mathcal{S}}^{-1}(A)\big) = \lambda(A)$.*

Proof. Since $\mathcal{G}_{\mathcal{S}} = \{B(n_1,\dots,n_p),\ (n_1,\dots,n_p) \in R_p(\mathcal{S}),\ p \geq 1\}$ is a generating sub-system of $\mathcal{B}_{(0,1)}$ (see (B) and (C)), the above equality holds if it does on elements of $\mathcal{G}_{\mathcal{S}}$. Now observe that if $(n_1,\dots,n_p) \in R_p(\mathcal{S})$ ((1), (2), (B)),

$$\Phi_{\mathcal{S}}^{-1}\big(B(n_1,\dots,n_p)\big) = \left(\prod_{i=1}^{p} \frac{B(n_1,\dots,n_i) - a_{n_1\dots n_{i-1}}}{b_{n_1\dots n_{i-1}} - a_{n_1\dots n_{i-1}}}\right) \times (0,1)^{\mathbb{N}}.$$

The proof of THEOREM 1.1. follows immediately.

Remark 1.1. *Let $\mathcal{U}$ denote the set of uniformly distributed sequences in $(0,1)$. In the case of Cantor series representation ($x = \sum_{i=1}^{+\infty} \frac{\varepsilon_i(x)}{q_1\dots q_i}$, $\varepsilon_k(x) \in \{0,\dots,q_k-1\}$, $Q = (q_k)_{k\geq 1} \in \big(\mathbb{N}^* \setminus \{1\}\big)^{\mathbb{N}^*}$), let $\Phi_Q = \Phi_{\mathcal{S}}$ where $\mathcal{S}$ is the number system associated with the Cantor series to "base" Q. Theorem 2 of [1] states that if on subsets of $\mathcal{U}$, we define $\lambda_Q(A) = \lambda(\Phi_Q(A))$ when $\Phi_Q(A) \in \mathcal{B}_{(0,1)}$, and take all possible sequences Q, then the only measure consistent with all the λ_Q on the corresponding subsets of $\mathcal{U}$ is λ_∞. If we observe (with notations from the proof of THEOREM 1.1.) that $\{\Phi_Q^{-1}(\mathcal{G}_Q),\ Q \in \big(\mathbb{N}^* \setminus \{1\}\big)^{\mathbb{N}^*}\}$ generates $\mathcal{B}_{\mathcal{U}}$, this statement results from THEOREM 1.1. and the fact that Φ_Q is onto.*

PROPOSITION 1.1. *For any $n \geq 0$, the random variable $t_n(\cdot)$ has uniform distribution, e.g. if $d \in (0,1)$, and $W_n(d) = \{x \in (0,1),\ t_n(x) \in (0,d)\}$, then $\lambda(W_n(d)) = d$.*

Proof. Clearly, given $(n_1,\dots,n_p) \in R_p(\mathcal{S})$, we have $\lambda\left(W_n(d) \cap B(n_1,\dots,n_p)\right) = d\lambda(B(n_1,\dots,n_p))$ ((1), (3)). With

$$\lambda(W_n(d)) = \sum_{(n_1,\dots,n_p)\in R(\mathcal{S})} \lambda(W_n(d) \cap B(n_1,\dots,n_p)),$$

and

$$\sum_{(n_1,\dots,n_p)\in R(\mathcal{S})} \lambda(B(n_1,\dots,n_p)) = 1,$$

the proof of PROPOSITION 1.1. follows.

Let us now introduce definitions for linear and logarithmic discrepancies associated to a sequence $u \in (0,1)^{\mathbb{N}}$ (cf [6]).

DEFINITION1.1. *Let* $u \in (0,1)^{\mathbb{N}}$, $p \geq 1$. *Let* $\mathcal{P}_p = \{(a_1,b_1) \times \ldots \times (a_p,b_p),\ 0 \leq a_i \leq b_i \leq 1,\ 1 \leq i \leq p\}$ *be the set of* p*-cubes in* $(0,1)^p$. *Let* $N \in \mathbb{N}^*$ *and* $P \in \mathcal{P}_p$*; then define*

$$\begin{cases} E_p(u,N,P) = \#\{i < N,\ u_i^{(p)} \in P\} - \lambda_p(P) \cdot N, \\ \\ D_p(u,N) = \sup_{P \in \mathcal{P}_p} |E_p(u,N,P)|, \\ \\ D_p(u) = \ \limsup_{N \to +\infty} \frac{1}{N} D_p(u,N). \end{cases}$$

$D_p(u)$ *is called the* p*-dimensional discrepancy (or linear discrepancy) of* u.

DEFINITION 1.2. *With the same notations as in the above definition, let*

$$D_p^{\ell}(u) = \limsup_{N \to +\infty} \frac{1}{(\log N)^p} D_p(u,N).$$

$D_p^{\ell}(u)$ *is called the* p*-dimensional logarithmic discrepancy of* u.

DEFINITION 1.3. *A sequence* $u \in (0,1)^{\mathbb{N}}$ *is completely uniformly distributed (abbreviate c.u.d.) if for any* $p \geq 1$, $D_p(u) = 0$.

Here are the criterions announced in the Introduction about discrepancies:

LEMMA 1.1. *Let* $u, v \in (0,1)^{\mathbb{N}}$. *If* $\frac{1}{N} \sum_{i<N} |u_i - v_i| \longrightarrow_{N \to +\infty} 0$, *then for all* $p \geq 1$

$$(D_p(u) = 0) \Leftrightarrow (D_p(v) = 0).$$

Proof. Using the Weyl criterion (see [6]), it is enough to prove for any $h \in \mathbb{Z}^p,\ h \neq (0, \ldots, 0)$, that if

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{j=1}^{N} e^{2i\pi <h, u_j^{(p)}>} \longrightarrow_{N \to +\infty} 0,$$

the limit relation still holds when u is replaced by v.

But for any $p \geq 1$, there exists a constant $C_p > 0$ such that for any $h \in \mathbb{Z}^p$ and $j \geq 1$

$$\left| e^{2i\pi <h, u_j^{(p)}>} - e^{2i\pi <h, v_j^{(p)}>} \right| \leq C_p. \max_{1 \leq t \leq p} |h_t| \cdot \left(\sum_{k=j}^{j+p-1} |u_k - v_k| \right).$$

The result then follows immediately, and the proof of LEMMA 1.1. is complete.

LEMMA 1.2. *Let* $u, v \in (0,1)^{\mathbb{N}}$ *be such that* $|u_m - v_m| = \mathcal{O}(e^{-m^2})$. *Then for all* $p \geq 1$,

$$D_p^{\ell}(u) = D_p^{\ell}(v).$$

Proof. Take notations from DEFINITION 1.1.. Let $P \in \mathcal{P}_p$ and $N \in \mathbb{N}^*$; then $\{i < N,\ u_i^{(p)} \in P\} \subset \{i < \sqrt{\log N}\} \cup \{\sqrt{\log N} \leq i < N,\ u_i^{(p)} \in P\}$. By the hypothesis, there exists $C > 0$ such that $|u_i - v_i| < Ce^{-i^2}$, $i \in \mathbb{N}$. If $i \geq \sqrt{\log N}$, then $e^{-i^2} < \frac{1}{N}$, and if $u_i^{(p)} \in P = \prod_{i=1}^{p} (a_i, b_i)$, then $v_i^{(p)} \in P' = \prod_{i=1}^{p} (a_i - \frac{C}{N}, b_i + \frac{C}{N})$. Thus we have

$$\#\{i < N,\ u_i^{(p)} \in P\} \leq 1 + \sqrt{\log N} + \#\{i < N,\ v_i^{(p)} \in P'\}.$$

One can find a constant $K > 0$ such that for N large enough, $N|\lambda(P') - \lambda(P)| \leq K$. Therefore there exists a constant $F_p > 0$ such that

$$E_p(u,N,P) \leq E_p(v,N,P') + F_p \sqrt{\log N} \leq D_p(v,N) + F_p \sqrt{\log N}.$$

Thus $D_p(u,N) \leq D_p(v,N) + F_p \sqrt{\log N}$. Since the problem is symmetrical in sequences u and v, we have obtained for N large enough,

$$|D_p(u,N) - D_p(v,N)| \leq F_p \sqrt{\log N}.$$

The proof of LEMMA 1.2. follows immediately.

2. λ-a.e. DISTRIBUTION OF $\big(t_n(.)\big)_{n\geq 0}$.

THEOREM 2.1. *Assume*

$$\lambda\left\{x,\ \frac{1}{N}\sum_{i=1}^{N}\frac{\lambda(B(n_1,\dots,n_{i+1}(x)))}{\lambda(B(n_1,\dots,n_i(x)))}\longrightarrow_{N\to+\infty} 0\right\}=1.$$

Then λ-a.e. the sequence $\big(t_n(x)\big)_{n\geq 0}$ is completely uniformly distributed.

Proof. Let

$$\Omega(\mathcal{S})=\left\{x,\ \frac{1}{N}\sum_{i=1}^{N}\frac{\lambda(B(n_1,\dots,n_{i+1}(x)))}{\lambda(B(n_1,\dots,n_i(x)))}\longrightarrow_{N\to+\infty} 0\right\},$$

$$\Omega^*(\mathcal{S})=\left\{x,\ \big(t_n(x)\big)_{n\geq 0}\text{ is c.u.d.}\right\}.$$

These sets belong to $\mathcal{B}_{(0,1)}$. Since $u\in\Phi_{\mathcal{S}}^{-1}(x)$ implies $|u_i-t_i(x)|\leq\frac{\lambda(B(n_1,\dots,n_{i+1}(x)))}{\lambda(B(n_1,\dots,n_i(x)))}$ (see (4)), it follows from LEMMA 1.1. that for any $p\geq 1$, $D_p\big((t_n(x))_{n\geq 0}\big)=D_p(u)$ when $x\in\Omega(\mathcal{S})$. Thence

$$\Phi_{\mathcal{S}}^{-1}\left(\Omega(\mathcal{S})\bigcap\Omega^*(\mathcal{S})\right)=\Phi_{\mathcal{S}}^{-1}\left(\Omega(\mathcal{S})\right)\bigcap\{u\in(0,1)^{\mathbb{N}},\ u\text{ is c.u.d.}\}.$$

$\Phi_{\mathcal{S}}^{-1}(\Omega(\mathcal{S}))$ has λ_∞-measure 1 by THEOREM 1.1., and $\Phi_{\mathcal{S}}^{-1}(\Omega(\mathcal{S})\bigcap\Omega^*(\mathcal{S}))$ has the same measure as $\Omega(\mathcal{S})\bigcap\Omega^*(\mathcal{S})$. The proof of THEOREM 2.1. follows.

Remark 2.1. *For Cantor series, the above criterion is hypothesis (H) in [10].*

THEOREM 2.2. *Assume*

$$\lambda\left\{x,\ \frac{\lambda(B(n_1,\dots,n_i(x)))}{\lambda(B(n_1,\dots,n_{i-1}(x)))}=\mathcal{O}(e^{-i^2}),\ i\geq 1\right\}=1.$$

Then λ-a.e., $D_p^\ell\big(\big(t_n(x)\big)_{n\geq 0}\big)=+\infty$, for any $p\geq 1$.

Proof. It is known (cf [6]) that for λ_∞ almost all sequences $u\in(0,1)^{\mathbb{N}}$, and for all $p\in\mathbb{N}$, $D_p^\ell(u)=+\infty$. The proof of THEOREM 2.2. runs along the same lines as the proof of the preceding theorem (using LEMMA 1.2. instead of LEMMA 1.1.) and is thus omitted.

3. APPLICATIONS.

3-1. (α,γ)-EXPANSIONS.

(α,γ)-expansions (cf [2]) are number systems $\mathcal{S}(\alpha,\gamma)=\big(]0,1],T_j,(J_{n,j})_{n\in\mathbb{N}^*}\big)_{j\geq 1}$ defined as follows: for any $j\geq 1$, let $\alpha_j:\mathbb{N}^*\to]0,1]$ be strictly decreasing, $\alpha_j(1)=1$, and $\lim_{n\to+\infty}\alpha_j(n)=0$. Let $\gamma_j:\mathbb{N}^*\to I\!R^+$ be such that $\gamma_j(n)\geq\alpha_j(n)-\alpha_j(n+1)$. Then let $J_{n,j}=]\alpha_j(n+1),\alpha_j(n)]$, $n\in\mathbb{N}^*$, and define $T_j(x)=\frac{x-\alpha_j(n+1)}{\gamma_j(n)}$ on $J_{n,j}$.

A necessary and sufficient condition for $\mathcal{S}(\alpha,\gamma)$ to satisfy condition (C) is that for any $(k_i)_{i\geq 1}\in R(\mathcal{S}(\alpha,\gamma))$, $\lim_{n\to+\infty}\left(\prod_{i=1}^{n-1}\gamma_i(k_i)\right)(\alpha_n(k_n)-\alpha_n(k_n+1))=0$ (cf [2]).

Cylinders are given by the following formula (cf [3])

$$B(k_1,\dots,k_p)=$$

$$\left]\sum_{j=1}^{p}\alpha_j(k_j+1)\prod_{m=1}^{j-1}\gamma_m(k_m),\ \sum_{j=1}^{p-1}\alpha_j(k_j+1)\prod_{m=1}^{j-1}\gamma_m(k_m)+\alpha_p(k_p)\prod_{m=1}^{p-1}\gamma_m(k_m)\right].$$

If $(k_i)_{i\geq 1} \in R(\mathcal{S}(\alpha,\gamma))$, then necessarily, for any $i \geq 1$,

$$\alpha_{i+1}(k_{i+1}+1) < \frac{\alpha_i(k_i) - \alpha_i(k_i+1)}{\gamma_i(k_i)}.$$

The associated expansion is the infinite series representation $x = \sum_{j\geq 1} \alpha_j(k_j+1) \prod_{i=1}^{j-1} \gamma_i(k_i)$. One obtains

$$\frac{\lambda(B(k_1,\ldots,k_{p+1}))}{\lambda(B(k_1,\ldots,k_p))} = \frac{\gamma_p(k_p)(\alpha_{p+1}(k_{p+1}) - \alpha_{p+1}(k_{p+1}+1))}{\alpha_p(k_p) - \alpha_p(k_p+1)}.$$

For Engel series, $\alpha_j(n) = \frac{1}{n}$, $\gamma_j(n) = \frac{1}{n+1}$. Let $\mathcal{S}(E)$ be the associated number system. Conditions on the digits are $k_{j+1} \geq k_j$, and since $\{x,\ \lim_{n\to+\infty} k_n(x) < +\infty\} = \mathbb{Q}\cap]0,1]$, from $\frac{\lambda(B(k_1,\ldots,k_{p+1}))}{\lambda(B(k_1,\ldots,k_p))} \leq \frac{1}{k_p}$ it follows that the criterion of THEOREM 2.1. applies. The corresponding result strengthens the one from [11, Chapter 11], since $t_n(x) = k_n(x)T^n(x)$ for $\mathcal{S}(E)$ and $T = T_j,\ j \geq 1$.

For Sylvester series, let $\mathcal{S}(S)$ be the number system associated to the choice $\alpha_j(n) = \frac{1}{n}$, $\gamma_j(n) = 1$. Conditions on the digits are $k_{j+1} \geq k_j(k_j+1)$. On the other hand, $\frac{\lambda(B(k_1,\ldots,k_{p+1}))}{\lambda(B(k_1,\ldots,k_p))} < \frac{1}{k_p^2}$. Since $k_1 \geq 2$, and $k_{j+1} \geq k_j^2$, it follows that the criterion of THEOREM 2.2. applies and strengthens the corresponding result of [11, Chapter 11] (here one has $k_n(x)(k_n(x)+1)T^n(x) = t_n(x)$).

For Lüroth series associated to the number system $\mathcal{S}(L)$ determined by the choice $\alpha_j(n) = \frac{1}{n}$, and $\gamma_j(n) = \frac{1}{n(n+1)}$, no conditions on the digits hold, and $\frac{\lambda(B(k_1,\ldots,k_{p+1}(x)))}{\lambda(B(k_1,\ldots,k_p)(x))}$ doesn't tend to zero on a set of full measure, nor does it's Cesaro mean. However, it can be seen that $(t_n(x), t_{n+1}(x)) \in]\frac{3}{4},1]\times[0,\frac{1}{2}[$ never occurs, thus the sequence $\big(t_n(x)\big)_{n\geq 1}$ is not c.u.d.. Notice that here, $t_n(.) = T^n$, where $T := T_j,\ j \geq 1$, and Remark 0.1. applies.

For (α,γ)-expansions, though in [11, Chapter 11] is used a result from [4] that states $\lambda(W_n(d)) = d + \mathcal{O}({\frac{5}{6}}^n)$, in our setting, this appears to be exact for Lüroth, Engel, and Sylvester series, since $(t_n(.))_{n\geq 1}$ is uniformly distributed (PROPOSITION 1.1.).

Proofs given in [11, Chapter 11], concerning uniform distribution of the relevant sequences, rely on a more general result from [8] (slightly modified in [11, Chapter 11]) which is used in the following form:

THEOREM. P. S. (W. Philipp, F. Schweiger). *Assume there exists a convergent series of positive real numbers, $\sum_{k\geq 1}\beta_k < +\infty$, such that for given $d \in]0,1]$ and $n,\ m \in \mathbb{N}^*$, the following inequality holds;*

$$\lambda(W_n(d) \cap W_{n+m}(d)) \leq$$

$$\lambda(W_n(d))\lambda(W_{n+m}(d)) + (\lambda(W_n(d)) + \lambda(W_{n+m}(d)))\beta_m + \lambda(W_{n+m}(d))\beta_n.$$

Then the sequence $\big(t_n(.)\big)_{n\geq 1}$ is λ-a.e. uniformly distributed.

We now give a criterion for that THEOREM P. S. may be applied:

THEOREM 3.1. *Assume there exists some constant $q > 1$ and a $k \in \mathbb{N}^*$ such that such that*

$$\lambda\left\{x,\ \forall p \in \mathbb{N}^*,\ \frac{\lambda(B(n_1,\ldots,n_{p+k}(x)))}{\lambda(B(n_1,\ldots,n_p(x)))} \leq \frac{1}{q}\right\} = 1.$$

Then the sequence $\big(t_n(.)\big)_{n\geq 1}$ is λ-a.e. uniformly distributed.

Remark 3.1. *THEOREM 3.1. holds for Cantor series and for many β-expansions (cf [9]). Notice that it holds for Lüroth series too, and since λ is invariant for the associated transformation, proves the ergodicity of the system.*

Proof of THEOREM 3.1. With hypothesis of THEOREM 3.1., and THEOREM. P. S., it is sufficient to show that for any $d \in (0,1)$

$$\lambda(W_n(d) \cap W_{n+m}(d)) \leq d^2 + \frac{d}{q^{\frac{m}{k}}}.$$

Inside $B(n_1, \ldots, n_p)$, for $p \geq N$, let $B(n_1, \ldots, n_{p+m}(d))$ be the cylinder of rank $p+m$ such that $a_{n_1 \ldots n_p}(1-d) + d b_{n_1 \ldots n_p} \in B(n_1 \ldots n_{p+m}(d))$ (see (B)). Put $d(n_1 \ldots n_p) = a_{n_1 \ldots n_p}(1-d) + d b_{n_1 \ldots n_p}$.

Next define

$$R_{p+m}\big(\mathcal{S}, n_1, \ldots, n_p, d\big) = \\ \{(n_{p+1}, \ldots, n_{p+m}) \,/ (n_1, \ldots n_{p+m}) \in R_{p+m}(\mathcal{S}),\ b_{n_1 \ldots n_{p+m}} < d(n_1 \ldots n_p)\}.$$

One can write the following:

$$\lambda(W_p(d) \cap W_{p+m}(d)) = \sum_{(n_1 \ldots n_p) \in R_p(\mathcal{S})} \lambda(W_p(d) \cap W_{p+m}(d) \cap B(n_1 \ldots n_p))$$

$$= \sum_{(n_1 \ldots n_p) \in R_p(\mathcal{S})} \left(\sum_{(n_{p+1} \ldots n_{p+m}) \in R_{p+m}\left(\mathcal{S}, n_1, \ldots, n_p, d\right)} d\lambda(B(n_1 \ldots n_{p+m})) \right)$$

$$+ \sum_{(n_1 \ldots n_p) \in R_p(\mathcal{S})} \lambda(W_p(d) \cap W_{p+m}(d) \cap B(n_1 \ldots n_{p+m}(d))).$$

$$\leq d^2 + \sum_{(n_1 \ldots n_p) \in R_p(\mathcal{S})} d \cdot \left(\frac{\lambda\big(B(n_1 \ldots n_{p+m}(d))\big)}{\lambda\big(B(n_1 \ldots n_p)\big)} \right) \cdot \lambda\big(B(n_1 \ldots n_p)\big)$$

$$\leq d^2 + d \cdot \frac{q}{q^{\frac{m}{k}}}$$

since the hypothesis implies $\frac{\lambda(B(n_1 \ldots n_{p+m})}{\lambda(B(n_1 \ldots n_p)} \leq \frac{q}{q^{\frac{m}{k}}}$. The proof of THEOREM 3.1. follows.

3-2. GENERALIZED CANTOR PRODUCTS.

Let $k \in \mathbb{N}^*$. In [7] the following number system is studied (call it $\mathcal{S}(C,k)$). Let $J_{n,j} = [\frac{n-1}{n+k-1}, \frac{n}{n+k}[$, and on $J_{n,j} := J_n$ let $T_j = T$ be defined by $T(x) = x \cdot \left(\frac{n+k}{n}\right)$. This is associated to the infinite product expansion $x = \prod_{i \geq 0} \frac{r_i(x)}{r_i(x)+k}$, where $r_i(x) = r$ if $T^i(x) \in J_{r_i(x)}$.

Conditions on the digits are $r_{i+1} \geq r_i^2 + (r_i - 1)(k-1)$, and $(r_i(x) = 1,\ i \geq 0) \Rightarrow (x = 0)$. In [7], the sequence $\big(t_n(.)\big)_{n \geq 0}$ is shown to be λ-a.e. c.u.d.. However, since $r_{i+1} \geq r_i^2$, and $\frac{\lambda(B(r_0, \ldots, r_{n+1}))}{\lambda(B(r_0, \ldots, r_n))} < \frac{1}{r_n^2}$, the conditions of THEOREM 2.2. and 2.1. are fulfilled, and the result follows.

3-3. CONTINUED FRACTIONS.

Let $[y]$ denote the integer part of the real number y. Let $\mathcal{S}(CF)$ be the system determined by the following choices; $T_j(x) = T(x) = \frac{1}{x} - \left[\frac{1}{x}\right]$ if $x \neq 0$, $T(0) = 0$, and $J_n =]\frac{1}{n+1}, \frac{1}{n}]$. This is associated to the continued fraction expansion $x = [0, a_1 \ldots a_n \ldots]$ (we take notations from [5]). Cylinders of rank n are the sets $B(a_1 \ldots a_n) = \left(\frac{p_n}{q_n}, \frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right)$, or $B(a_1 \ldots a_n) = \left(\frac{p_n+p_{n-1}}{q_n+q_{n-1}}, \frac{p_n}{q_n}\right)$ depending on the parity of n. One can compute that

$$\begin{cases} t_n(x) = \frac{(q_n+q_{n-1})\cdot T^n x}{q_n+q_{n-1}\cdot T^n x}, & n \equiv 0(2) \\ t_n(x) = \frac{q_n(1-T^n x)}{q_n+q_{n-1}\cdot T^n x 0}, & n \equiv 1(2) \end{cases}$$

Using ergodicity of the transformation T with respect to it's invariant measure μ, determined by it's density $d\mu = \frac{d\lambda}{(1+x)\log 2}$, one can show that the criterion of THEOREM 2.1. fails since the set involved has measure 0. However, THEOREM 3.1. may be applied to obtain uniform distribution since $\frac{\lambda(B(a_1 \ldots a_{n+1}))}{\lambda(B(a_1 \ldots a_n))} \leq \frac{1}{2}$ (cf [5, p. 58-59]).

As in the case of Lüroth series, one can directly prove that the sequence $\left(t_n(x)\right)_{n\geq 0}$ is never c.u.d., since the event $(t_n(x), t_{n+1}(x), t_{n+2}(x)) \in]\frac{1}{4}, \frac{1}{3}[^3$ never occurs: indeed, with formulas from [5, p. 59], one obtains $\left(n \text{ odd and } t_n(x) \in]\frac{1}{4}, \frac{1}{3}[\right) \Rightarrow \left(t_{n+1}(x) > \frac{1}{3}\right)$.

PROBLEMS. *Is there a number system $\mathcal{S}$ for which criterion of THEOREM 2.1. fails, though the sequence $\left(t_n(.)\right)_{n\geq 0}$ is λ-a.e. c.u.d.?*

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