MULTIPLICITY, RANK PAIRS

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Abstract. For any pair (m, r) such that $2 \leq m \leq r < \infty$, we construct an ergodic dynamical system having spectral multiplicity m and rank r. The essential range of the multiplicity function is described. If $r \geq 2$, the pair (m, r) also has a weakly mixing realization.

Contents

1991 *Mathematics Subject Classification*: Primary 28D05,54H20

Key words and phrases. Multiplicity, rank, compact group extension, Morse automorphism, weakly mixing, covering number.

◇ Supported by the KBN grant no 2 P30103107. The author acknowledges the hospitality of Mathematics Department of Université de Bretagne Occidentale, Brest, where this paper was written.

♡ Supported by C.A.F. Nord Finistère.

0. Introduction.

Given a dynamical system (X, μ, T) one associates to it a measure theoretic invariant, the rank $r(T)$, and a spectral invariant, the multiplicity $m(T)$. The pair $(m(T), r(T))$ is such that $1 \leq m(T) \leq r(T) \leq \infty$.

It was conjectured in [M1] that for any such pair of integers (or ∞), there exists an ergodic system (X, μ, T) realizing it. Pairs $(1,1)$ where constructed in [Ch], $(1,2)$ in [dJ], $(1,r)$ in [M1], $(2,r)$ in [GoLe], (r,r) in [R1,2], $(r,2r)$ in [M2], $(p-1,p)$ in [FeKw] and $(1,\infty)$ in [LeSi], [Fe]. Gaussian-Kronecker systems allways realize the pair $(1,\infty)$ ([dlR]). The latest result of this series is a density theorem [FeKwMa] proving that given m, the set of r's such that the pair (m,r) is realizable has density 1.

In this note we construct realizations of pairs (m,r) with $2 \leq m \leq r < \infty$. The pair (∞, ∞) however is realized with any ergodic system of positive entropy. Thus, together with [M1], we prove that all pairs (m,r) with $1 \leq m \leq r < \infty$ are obtainable.

The transformations we use are continuous Morse automorphisms over a finite abelian group (see **I**. for the preliminaries). These systems however have partly discrete spectrum.

However the same pairs (m,r) can be obtained within the class of weakly mixing systems. We give to this end some hints in Remark IV.1., but for the sake of simplicity, computations are only carried out with full details for the Morse automorphisms.

Our examples (see **II**.) sit in the class of so called *natural factors* of a compact abelian group extension ([KwJLe]).

We first construct an ergodic group extension $(X \times G, T_\varphi, \mu \otimes m_G)$ realizing the pair (r,r). Next we produce a natural factor $(X \times H, T_{\varphi_H}, \mu \otimes m_H)$, where $H = G/H_0$ is a quotient group.

Using methods developed in [KwJLe] we compute the spectral multiplicities of the systems: the ones of the natural factor decrease. But surprisingly the ranks of the natural factors remain equal to the rank of the initial system, r.

The systems have a continuous Morse shift representation which we use for the rank computations. In Section **III**. we prove the following for $2 \leq r < \infty$:

Theorem (r,r)**.** *The system $(X \times G, T_\varphi, \mu \otimes m_G)$ satisfies $r(T_\varphi) = r = m(T_\varphi)$, and the essential range of its multiplicity function is $\{d : d|r\}$. It is measure theoretically isomorphic to a strictly ergodic continuous Morse automorphism.*

If $2 \leq m < r < \infty$, passing to the natural factor $(X \times H, T_{\varphi_H}, \mu \otimes m_H)$, we prove in section **III**. that $m(T_{\varphi_H}) = m$ while in section **IV**. we prove that its rank is r, by proving that its covering number satisfies $F^\star(T_{\varphi_H}) < \frac{1}{r-1}$. We obtain:

Theorem (m,r)**.** *The system $(X \times H, T_{\varphi_H}, \mu \otimes m_H)$ is such that $r(T_{\varphi_H}) = r$, $m(T_{\varphi_H}) = m$, and the essential range of its multiplicity function is $\{1, \ldots, m\}$. It is measure theoretically isomorphic to a strictly ergodic continuous Morse automorphism.*

I. Preliminaries.

Throughout this paper G shall denote an additive abelian finite group, and Ω the space of bi-infinite sequences taking values in G.

I.1. Blocks and operations on blocks.

A finite sequence $B = B[0]...B[k-1]$, $B[i] \in G$, $k \geq 1$, is called a block over G. The number k is called the length of B and denoted $|B|$. If $\omega \in \Omega$ (or ω is a one-sided sequence over G) and B is a block then $\omega[i,s]$ and $B[i,s]$ $(0 \leq i \leq s \leq k-1)$ denote the blocks $\omega[i]...\omega[s]$ and $B[i]...B[s]$ respectively. If $C = C[0]...C[f-1]$ is another block then the *concatenation* of B and C is the block

$$BC = B[0]...B[k-1]C[0]...C[f-1].$$

Concatenation extends to more than two blocks in the obvious way. We define also for $q \in \mathbb{Z}$,

$$B^q = \overbrace{B\ldots\ldots B}^{q \text{ times}}.$$

If $v : G \to G$ is a *group automorphism*, let $v(B)$ be the block

$$v(B) = v(B[0])...v(B[k-1]).$$

If $g \in G$, by $B(g)$, we will denote the block $B+g = B(g) = (B[0]+g)...(B[k-1]+g)$. Then $v(B(g)) = v(B)(v(g)),\ g \in G$. Finally, we define the product $B \times C$ of B and C as follows $(|C| = f)$:

$$B \times C = B(C[0])...B(C[f-1]).$$

As for concatenation, this multiplication operation "$\times$" is extended to more than two blocks, and is associative.

I.2. Occurrences, frequencies, density, $\bar{d}$ distance.

The block B is said to occur at place i in ω (resp. in C as above $(k \leq f)$) if $\omega[i, i+|B|-1] = B$ (resp. $C[i, i+|B|-1] = B$). We shall write $B \lessdot \omega$ (resp. $B \lessdot C$) when this happens for some position i.

The frequency of B in C (resp. in ω) is the number

$$fr(B,C) = |C|^{-1}\#\{0 \leq i \leq |C|-|B|-1;\ B \text{ occurs at place } i \text{ in } C\},$$

$$\text{(resp. } fr(B,\omega) = \lim_{s\to\infty} fr(B, \omega[0, s-1]) \text{ if this limit exists).}$$

For a one sided infinite subsequence of ω, $E = \{\omega[n], n \in I \subset \mathbb{N}\}$, we call the density of E the corresponding density of the set I in $\mathbb{N}$, and denote it by $D(E,\omega)$ (if it exists).

Let $\delta > 0$. We say that B δ-occurs at place i in C (resp. in ω) if

$$\bar{d}(B, C[i, i+|B|-1]) < \delta \quad \text{(resp. } \bar{d}(B, \omega[i, i+|B|-1]) < \delta),$$

where

$$\bar{d}(x_1...x_n, y_1...y_n) = n^{-1}\#\{i : x_i \neq y_i\};$$

$\bar{d}$ is the *normalized Hamming distance or d-bar distance* between blocks. It has the following properties:

$$(1)\quad \begin{cases} (a) & \bar{d}(B(g), C(g)) = \bar{d}(B,C),\ g \in G, \\ (b) & \bar{d}(v(B), v(C)) = \bar{d}(B,C), \\ (c) & \bar{d}(B \times C, B \times D) = \bar{d}(C,D), \\ (d) & \bar{d}(A_1 \ldots A_k, B_1 \ldots B_k) = \frac{1}{k}\sum_{i=1}^{k} \bar{d}(A_i, B_i) \\ & \qquad (|A_i| = |B_j|,\ 1 \leq i,j \leq k), \\ (e) & \bar{d}(A_1A_2, B_1B_2) = \frac{|A_1|}{|A_1|+|A_2|}\bar{d}(A_1,B_1) + \frac{|A_2|}{|A_1|+|A_2|}\bar{d}(A_2,B_2) \\ & \qquad (|A_i| = |B_i|,\ i = 1,2), \\ (f) & \bar{d}(A,B) \geq \frac{|A_1|}{|A|}\bar{d}(A_1,B_1) \ \ if \ \ A = A_1A_2,\ B = B_1B_2, \\ (g) & \bar{d}(B,C) \leq \bar{d}(B,A) + \bar{d}(A,C). \end{cases}$$

If $|B| \geq 2$, we let $\check{B}$ be the block of length $|B|-1$ defined by

$$\check{B}[i] = B[i+1] - B[i],\ 0 \leq i \leq |B|-2. \tag{2}$$

We shall also use the following property of the $\bar{d}$ distance;

$$\bar{d}(\check{B}, \check{C}) \leq 3\bar{d}(B, C). \tag{3}$$

I.3. The dynamical system associated to a sequence.

We let S denote the left shift homeomorphism of Ω or $\Omega^{\square} = (G \cup \{\square\})^{\mathbb{Z}}$. If $\omega = \omega[0]\omega[1]\dots$ is a one-sided sequence over G, we let $\omega_{\square}$ be the element of $\Omega^{\square}$ defined by $\omega_{\square}[n] = \omega[n]$ if $n \geq 0$, $\omega_{\square}[n] = \square$ otherwise. We then define

$$\Omega_\omega = \{y \in \Omega : \exists (n_i),\ n_i \to \infty,\ y = \lim_i S^{n_i}\omega_{\square}\}.$$

The topological flow (Ω_ω, S) is *minimal* if there is no proper closed and S-invariant subset of Ω_ω. We say that (Ω_ω, S) is *uniquely ergodic* if there is a unique borelian normalized S-invariant measure μ_ω on Ω_ω. (Ω_ω, S) is said to be *strictly ergodic* if it is both minimal and uniquely ergodic (for short, we say that ω is strictly ergodic).

If ω is strictly ergodic, then for each block B, and $q \in \mathbb{Z}$,

$$\mu_\omega([B]_q) = fr(B, \omega),$$

where $[B]_q = \{y \in \Omega_\omega : y[q, q+|B|-1] = B\}$.

I.4. Rank and covering number of $(\Omega_\omega, S, \mu_\omega)$.

For an ergodic dynamical system, the rank and the covering number are classical measure theoretic invariants ([dJ], [Fe]). In the case of a symbolic strictly ergodic system $(\Omega_\omega, S, \mu_\omega)$, we formulate their "combinatorial" definitions bellow.

Let $\mathcal{A}$ be a (finite) family of blocks and B a block such that $|B| \in \{|A| : A \in \mathcal{A}\}$, we let

$$\bar{d}(B, \mathcal{A}) = \min\{\bar{d}(B, A) : A \in \mathcal{A},\ |A| = |B|\}.$$

If $\mathcal{A} = \{A_1, \dots, A_k\}$, C is a block, and $\delta > 0$, we define

$$t_\delta(\mathcal{A}, C) = t_\delta(A_1, \dots, A_k, C) = \max\{\frac{|C_1| + \dots + |C_p|}{|C|}\},$$

where the maximum is taken over all concatenations of the form

$$C = \epsilon_1 C_1 \epsilon_2 \dots \epsilon_p C_p \epsilon_{p+1}$$

for which $\bar{d}(C_i, \mathcal{A}) < \delta$, $1 \leq i \leq p$. Then we define, for a strictly ergodic one-sided sequence ω,

$$t_\delta(\mathcal{A}, \omega) = \liminf_{N \to \infty} t_\delta(\mathcal{A}, \omega[0, N]) (= \lim_{N \to \infty} t_\delta(\mathcal{A}, \omega[0, N])).$$

In particular, $t_\delta(A, \omega)$ is defined for a block A. It is known ([dJ], [M2]) that in the case under consideration the *rank of* $(\Omega_\omega, S, \mu_\omega)$ *is at most* r if for any $\delta > 0$ and any $N \in \mathbb{N}$, there exists $\mathcal{A}$ of cardinality r such that $|A| \geq N$, $A \in \mathcal{A}$, and

$$t_\delta(\mathcal{A}, \omega) \geq 1 - \delta.$$

Then $(\Omega_\omega, S, \mu_\omega)$ is of *rank equal to* r if it is of rank at most r but not at most $r-1$. The rank is a measure theoretic invariant. We denote it by $r(S)$ or r_ω.

We say that the *covering number* $F^\star(\omega)$ (also denoted by $F^\star(S)$) of ω is at least a $(0 < a < 1)$ if

$$\forall\ \delta > 0,\ \forall\ n \geq 1,\ \exists\ A,\ |A| \geq n,\ t_\delta(A, \omega) \geq a.$$

Then the *covering number* $F^\star(\omega)$ is the supremum of such a's. The covering number is a measure theoretic invariant, and

$$r_\omega \cdot F^\star(\omega) \geq 1.$$

I.5. Adding machines and cocycles.

Let $T : (X, B, \mu) \longrightarrow (X, B, \mu)$ be an (n_t)*-adic adding machine*, *i.e.* $n_t | n_{t+1}$, $\lambda_{t+1} = n_{t+1}/n_t \geq 2$ for $t \geq 0,\ \lambda_0 = n_0 \geq 2$,

$$X = \{x = \sum_{t=0}^{\infty} q_t n_{t-1} :\ 0 \leq q_t \leq \lambda_t - 1, n_{-1} = 1\}$$

is the *group of* (n_t)*-adic numbers* and $Tx = x + \hat{1},\ \hat{1} = (1, 0, 0, ...)$. The space X has the *standard sequence* (ξ_t) *of* T*-towers.* Namely,

$$\xi_t = (D_0^t, ..., D_{n_t-1}^t),$$

where

$$D_0^t = \{x \in X :\ q_0 = ... = q_t = 0\},\ T^s(D_0^t) = D_s^t,\ s = 0, ..., n_t - 1.$$

Then ξ_{t+1} refines ξ_t and the sequence of partitions (ξ_t) converges to the point partition.

A *cocycle* is a measurable function $\varphi : X \to G$. A cocycle φ defines an *automorphism* T_φ *on* $(X \times G, \mu \otimes m_G)$ by

$$T_\varphi(x, g) = (Tx, g + \varphi(x)),\ x \in X,\ g \in G,$$

where m_G is the Haar measure of G. T_φ is ergodic iff for every non-trivial $\gamma \in \hat{G}$ ($\hat{G}$ is the dual group), there is no measurable solution $f : X \to S^1$ to the functional equation

$$\gamma(\varphi(x)) = f(Tx)/f(x),\ x \in X \qquad \text{[Pa]}.$$

I.6. Morse cocycles (M-cocycles).

We say $\varphi : X \to G$ is an *M-cocycle* if for every $t \geq 0$, φ is constant on each level D_i^t for $i = 0, 1, ...,\ n_t - 2$. Such a cocycle is defined by a sequence of blocks $(A_t)_{t \geq 0}$, $|A_t| = n_t - 1$, and

$$\varphi_{|D_i^t} = A_t[i],\ i = 0, 1, ..., n_t - 2.$$

Now we describe *Morse sequences (M-sequences)*. Let $b_0, b_1, ...$ be finite blocks with $|b_t| = \lambda_t$, $b_t[0] = 0$, $t \geq 0$. Then we may define a one-sided sequence by

$$\omega = b_0 \times b_1 \times ...$$

Such a sequence is called a *generalized Morse sequence* over G if it is not periodic and if each of the sequences

$$\omega_t = b_t \times b_{t+1}, ..., \ t \geq 0,$$

contains every symbol in G. By grouping some of the b_t's we can assume that each block b_t contains every symbol in G. When this is the case, it is known [Ma] that (Ω_ω, S) is strictly ergodic if $fr(g, b_t) = \frac{1}{\#G}$ for every $g \in G$ and $t \geq 0$. It is not hard to observe that the condition

$$fr(g, b_t) \geq \rho > 0, \ for\ every\ t \geq 0, \ g \in G,$$

already implies the strict ergodicity of ω.

A Morse sequence ω allows one to define an M-cocycle $\varphi = \varphi_\omega$ on X as follows: let

$$B_t = b_0 \times ... \times b_t, \ t \geq 0.$$

Then choose $(A_t)_{t\geq 0} = (\check{B}_t)_{t\geq 0}$ (cf. (2)): it is easy to check that it defines an M-cocycle φ as above.

It follows from [Kw] and [Le] that the dynamical systems $(\Omega_\omega, S, \mu_\omega)$ and $(X \times G, T_\varphi, \mu \otimes m_G)$ are measure theoretically isomorphic if ω is strictly ergodic. We finally define

$$(4) \qquad \begin{cases} \varphi^{(p)}(x) = \varphi(x) + \ldots + \varphi(T^{p-1}x), \ x \in X, \ p \geq 1. \\ \text{Then } \ \varphi^{(p)}_{|D_i^t} = B_t[i+p] - B_t[i], \ p \geq 1, \ 0 \leq i < n_t - p - 1. \end{cases}$$

I.7. Spectral multiplicity(ies) and continuous Morse sequences.

If ω is a strictly ergodic Morse sequence over G, and $\varphi = \varphi_\omega$ is the associated M-cocycle, for $\gamma \in \hat{G}$, we define

$$L_\gamma = \{f \otimes \gamma \in L^2(X \times G, \mu \otimes m_G); \ f \in L^2(X, \mu)\}.$$

Let U_{T_φ} be the unitary operator induced by T_φ on $L^2(X \times G, \mu \otimes m_G)$. The subspaces L_γ are U_{T_φ}-invariant and we have the spectral decomposition

$$L^2(X \times G, \mu \otimes m_G) = \bigoplus_{\gamma \in \hat{G}} L_\gamma.$$

It is shown in [KwSi] that T_φ has simple spectrum on each L_γ. Let μ_γ be the spectral measure of T_φ on L_γ. It follows from [Ke] that any two of those μ_γ are either orthogonal or equivalent.

The *essential range of the multiplicity function* is then the subset of $\mathbb{N}$ consisting of the cardinalities of the equivalence classes of the measures $\mu_\gamma, \ \gamma \in \hat{G}$. It is a *spectral invariant*. The *spectral multiplicity* then coincides with the maximal number of this set. We denote it by $m(T_\varphi)$.

The subspace $L_{\hat{e}}$ ($\hat{e} :=$ the trivial character) is generated by the eigenfunctions of T_φ corresponding to all n_t-roots of unity. A Morse sequence ω is *continuous* (or the M-cocycle φ_ω is *weakly mixing*) if $L_{\hat{e}}$ contains all eigenfunctions of T_φ, or equivalently if each measure $\mu_\gamma, \ \gamma \neq \hat{e}$, is continuous. The following is proved in [IwLa]:

Proposition I.0. *For a strictly ergodic M-sequence ω over G, a sufficient condition for it to be continuous is that $\#G \big| \lambda_t$, $t \geq 0$.*

I.8. Preliminary results.

We know that the measures $\mu_\gamma, \gamma \in \hat{G}$, are either equivalent or orthogonal. We shall use the following criterions to know which case holds:

Proposition I.1 ([FeKw], [GoKwLeLi]). *If $\omega = b_0 \times b_1 \times ...$ is a strictly ergodic Morse sequence, where for every t, b_t is of the form*

$$b_t = d_t v(d_t) ... v^{k_t - 1}(d_t),$$

where v is an automorphism of G, d_t are blocks and k_t are integers such that $\sum_{t=0}^{\infty} \frac{1}{k_t} < \infty$, then $\mu_\gamma \simeq \mu_{\hat{v}(\gamma)}$ for all γ in *$\hat{G}$, where $\hat{v}$ is the dual automorphism to v.*

Proposition I.2 ([GoKwLeLi]). *Additionally, if for given $\gamma, \gamma' \in \hat{G}$,*

$$\lim_{t\to\infty} \int \gamma[\varphi^{(n_t)}(x)] d\mu \text{ and } \lim_{t\to\infty} \int \gamma'[\varphi^{(n_t)}(x)] d\mu \quad \text{(see (4))}$$

exist and differ from each other, then $\mu_\gamma \perp \mu_{\gamma'}$ (note that $T^{n_t} \to Id_X$ in the weak topology).

I.9. Quotient M-cocycle (sequence).

Let H_0 be a subgroup of G, and $H = G/_{H_0}$ be the quotient group. Let $\pi_H : G \to H$ be the quotient map (a group homomorphism). If m_H denotes the Haar measure on H, then the map

$$P_H := Id_X \times \pi_H : (X \times G, T_\varphi, \mu \otimes m_G) \to (X \times H, T_{\varphi_H}, \mu \otimes m_H)$$

is a factor map (the *natural factor map* associated to the quotient group H).

For a block B over G, we define the block B_H over H by

$$B_H = \pi_H(B[0]) \dots \pi_H(B[|B| - 1]),$$

and for $\eta \in \Omega$, let η_H be defined by

$$\eta_H[n] = \pi_H(\eta[n]), \quad n \in \mathbb{Z}.$$

Using the obvious equality $\pi_H(B \times C) = \pi_H(B) \times \pi_H(C)$, it is not hard to see that if $\omega = b_0 \times b_1 \times \dots$ is a M-sequence over G (we let φ be the corresponding M-cocycle), then $\omega_H = b_{0_H} \times b_{1_H} \times \dots$ and it is an M-sequence, over H. It determines an M-cocycle φ_H which moreover satisfies

$$\varphi_H = \pi_H \circ \varphi.$$

We now recall some facts from [KwJLe]: define $L_{\gamma,H} = \{f \otimes \gamma : f \in L^2(X, \mu)\}$ for $\gamma \in \hat{H}$. Then using the factor map P_H, we can identify this $U_{T_{\varphi_H}}$-invariant subspace of $L^2(X \times H, \mu \otimes m_H)$ with the U_{T_φ}-invariant subspace $L_{\tilde{\gamma}}$ of $L^2(X \times G, \mu \otimes m_G)$ defined in **I**.7., where $\tilde{\gamma} \in \hat{G}$ is the unique element of $\hat{G}$ such that

$$\tilde{\gamma}_{|H_0} = \hat{e}_{|H_0} \text{ and } \gamma \circ \pi_H = \tilde{\gamma}.$$

In fact if $V_{P_H} : L_{\tilde{\gamma}} \to L_{\gamma,H}$ is defined by $V_{P_H}(f \otimes \tilde{\gamma}) = f \otimes \gamma$ then it provides a unitary equivalence between the pairs $(L_{\tilde{\gamma}}, U_{T_\varphi})$ and $(L_{\gamma,H}, U_{T_{\varphi_H}})$. The cyclicity of the spaces $L_{\gamma,H}$ under the related unitary action remains, as in **I**.7..

So we obtain

Proposition I.3. *If the assumptions of Proposition I.1. hold, and $\gamma, \gamma' \in \hat{H}$ are such that $\tilde{\gamma}, \tilde{\gamma}'$ belong to a same $\hat{v}$-trajectory, then $\mu_\gamma \simeq \mu_{\gamma'}$.*

II. The candidates for (m, r).

First of all we select $G = \mathbb{Z}_n^r$ (direct product) and $r < n$ ($r \geq 2$ is given). Some conditions on n shall be specified in the proof of Theorem (m, r) (see **IV**.3).

For $h \in G$, we write $h = (h_1, \dots, h_r)$, $h_i \in \mathbb{Z}_n$. We let

$$e_i = (\underbrace{0, \dots, 0}_{i-1 \text{ times}}, 1, 0, \dots, 0), \quad 1 \leq i \leq r.$$

Then $h = h_1 e_1 + \ldots + h_r e_r$. Let $v \in Aut(G)$ be defined by

$$v(h) = (h_r, h_1, \dots, h_{r-1}), \quad h \in G.$$

Now we start defining some blocks (the goal is to define the blocks $b_t,\ t \geq 0$). Put, for $t \geq 0$,

$$\begin{cases} F_1 &= & F_{t,1} &= & 0e_1(2e_1)\ldots((l-1)e_1), \quad l = l_t, \\ F_2 &= & F_{t,2} &= & 0e_2(2e_2)\ldots((l-1)e_2), \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_r &= & F_{t,r} &= & 0e_r(2e_r)\ldots((l-1)e_r), \end{cases}$$

where $n \big| l_t$ and $l_t \to \infty$. Observe that $|F_i| = l_t$, and that $v(F_i) = F_{i+1}$, $1 \leq i \leq r$, where $F_{r+1} := F_1$. Let

$$\begin{cases} \beta_1 &= & \beta_{t,1} &= & F_1 \times F_2 \times \ldots \times F_r, \\ \beta_2 &= & \beta_{t,2} &= & F_2 \times \ldots \times F_r \times F_1, \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_r &= & \beta_{t,r} &= & F_r \times F_1 \times \ldots \times F_{r-1}. \end{cases}$$

Then $|\beta_i| = l^r$, and $v(\beta_i) = \beta_{i+1}$, $1 \leq i \leq r$, where similarly $\beta_{r+1} := \beta_1$. Next let

$$\beta = \beta_t = \beta_1 \beta_2 \ldots \beta_r \ \ \text{(concatenation)} \ \ (= \beta_1 v(\beta_1) \ldots v^{r-1}(\beta_1)),$$

and

$$b_t = \beta_t^{p_t} = \overbrace{\beta \ldots\ldots \beta}^{p_t \text{ times}} = \beta_1 v(\beta_1) \ldots\ldots v^{k_t - 1}(\beta_1),$$

where $k_t = rp_t$. We have $|b_t| = rp_t l_t^r := \lambda_t$. We choose the sequence (p_t) such that $\sum_{t \geq 0} \frac{1}{p_t} < \infty$. Finally we let $n_t = \lambda_0 \ldots \lambda_t$ and $B_t = b_0 \times \ldots \times b_t$ as in **I**.6.. Let ω be the associated M-sequence over G and φ the related M-cocycle. Notice that $n^r \big| \lambda_t$.

Now given $2 \leq m < r$, we define

$$\begin{cases} H_0 = \{0\}^m \times \mathbb{Z}_n^{r-m}, \\ H = G \big/ H_0 \equiv \mathbb{Z}_n^m, \end{cases}$$

and let the subscript H denote the corresponding blocks, M-sequence, M-cocycle as in **I**.9..

Proposition II.1. *The sequence ω is strictly ergodic and $(\Omega_\omega, S, \mu_\omega)$ is measure theoretically isomorphic to $(X \times G, T_\varphi, \mu \otimes m_G)$. Moreover it is a continuous M-sequence.*

Proof. Using **I**.6., we shall prove that for $t \geq 0$,

$$fr(g, b_t) = \frac{1}{n^r}.$$

Looking more closely to the blocks b_t it is seen that for $g \in G$, $fr(g, \beta_{t,i}) = \frac{1}{\#G} = \frac{1}{n^r}$, $1 \leq i \leq r$. So $fr(g, \beta_t) = \frac{1}{n^r}$ and $fr(g, b_t) = \frac{1}{n^r}$.

Since $n^r \big| \lambda_t$, we may apply Proposition I.0. to obtain the continuity. ■

Proposition II.2. *The sequence ω_H is strictly ergodic and $(\Omega_{\omega_H}, S, \mu_{\omega_H})$ is measure theoretically isomorphic to $(X \times H, T_{\varphi_H}, \mu \otimes m_H)$. Moreover it is a continuous M-sequence.*

Proof. We use facts from **I**.6. again. The strict ergodicity of ω_H follows, as in the preceding proof, from the equation

$$fr(h, b_{t_H}) = \sum_{\pi_H(g)=h} fr(g, b_t) = \frac{\#H_0}{n^r} (= \frac{1}{\#H} = \frac{1}{n^m}), \quad h \in H.$$

The continuity is proved as for ω. ■

The system $(\Omega_\omega, S, \mu_\omega)$ is our candidate for the (r, r) pair, while *the system $(\Omega_{\omega_H}, S, \mu_{\omega_H})$ is our candidate for the (m, r) pair.*

III. Spectral multiplicity(ies) of T_φ, T_{φ_H}, Theorem (r, r).

We first apply Propositions I.1., II.1., **I**.7. and the description of candidates in **II**. to obtain, using $\sum_{t \geq 0} \frac{1}{k_t} < \infty$,

Proposition III.1. *One has $m(T_\varphi) \geq r$.*

Proof. As indicated above, only check that the v-trajectory of e_1 is of length r. ■

Proposition III.2. *One has $m(T_{\varphi_H}) \geq m$.*

Proof. Let $\mathcal{H}_0 = \text{ann } H_0 = \{\gamma \in \hat{G} : \gamma_{|H_0} = \hat{e}_{|H_0}\}$. Then as in [KwJLe], using **I**.9. and Proposition I.3., we have

$$m(T_{\varphi_H}) \geq \max\{\#(\mathcal{H}_0 \cap \Upsilon)\},$$

where Υ runs along the set of $\hat{v}$-trajectories in $\hat{G}$. As is easily computed, this max is equal to m. ■

To prove that $m(T_\varphi) = r$, we shall use Proposition I.2..

Proposition III.3. *One has $m(T_\varphi) = r$.*

Proof. Recall that from the block constructions in **II**. and **I**.6., we have $\varphi_{|D_i^t} = B_t[i+1] - B_t[i]$. For $h = h_1 e_1 + \ldots + h_r e_r \in G$, let $\bar{h}(t) = h_1 + h_2 l_{t+1} + \ldots + h_r l_{t+1}^{r-1}$. Then $\bar{h}(t) < 2n l_{t+1}^{r-1}$, hence $\bar{h}(t) \,/\, \lambda_{t+1} \to 0$. Moreover,

$$\varphi^{(\bar{h}(t) n_t)} = b_{t+1}[p + \bar{h}(t)] - b_{t+1}[p], \ x \in D_{pn_t + j'}^{t+1}$$

for $0 \le j' < n_t$ and $0 \le p \le \lambda_{t+1} - \bar{h}(t) - 2$. We write $p = qrl_{t+1}^r + il_{t+1}^r + w$, where $0 \le q < p_{t+1}$, $0 \le i < r$, $0 \le w < l_{t+1}^r$. Next we write $w = w_1 + w_2 l_{t+1} + \ldots + w_r l_{t+1}^r$ where $0 \le w_j < l_{t+1}$, $1 \le j \le r$. Assuming that for each $1 \le j \le r$, $w_j < l_{t+1} - n - 1$, we see from the block construction that for such p's,

$$\varphi^{(\bar{h}(t)n_t)} = v^{i-1}(\beta_{t+1,1}[w + \bar{h}(t)] - \beta_{t+1,1}[w]) = v^{i-1}(h),\ x \in D^{t+1}_{pn_t+j'}.$$

For given h and t, the set of p's satisfying the desired conditions from above has cardinality $\lambda_{t+1} - o(\lambda_{t+1})$, hence we obtain, decomposing X along each T-tower ξ_{t+1}, that for any $\gamma \in \hat{G}$,

$$\lim_{t\to\infty} \int_X \gamma(\varphi^{(\bar{h}(t)n_t)}(x))d\mu(x) = \frac{1}{r}\sum_{i=0}^{r-1} \hat{v}^i(\gamma)(h).$$

Moreover, $(\bar{h}(t)n_t)$ is a rigid time for T. For $\gamma" \in \hat{G}$, let $A_{\gamma"} = \frac{1}{r}\sum_{i=0}^{r-1} \hat{v}^i(\gamma")$. If $\gamma, \gamma' \in \hat{G}$ do not belong to the same $\hat{v}$-trajectory, then $A_\gamma \perp A_{\gamma'}$ (in $L^2(G, m_G)$). Hence since $A_\gamma \not\equiv 0$, there exists an $h \in G$ such that $A_\gamma(h) \ne A_{\gamma'}(h)$. Then taking the rigid time $(\bar{h}(t)n_t)$, applying Proposition II.2., we obtain $\mu_\gamma \perp \mu_{\gamma'}$. We conclude with Proposition III.1.. ■

Proposition III.4. *One has* $m(T_{\varphi_H}) = m$.

Proof. Using the same notations as in **I**.9. and Propositions II.2., III.2., III.3., we deduce that if $\gamma, \gamma' \in \hat{H}$, if $\mu_{\tilde{\gamma}} \perp \mu_{\tilde{\gamma}'}$ then $\mu_\gamma \perp \mu_{\gamma'}$ as in [KwJLe]. Hence we may deduce the equality

$$m(T_{\varphi_H}) = \max\{\#(\mathcal{H}_0 \cap \Upsilon)\},$$

where Υ runs along the set of $\hat{v}$-trajectories in $\hat{G}$. ■

Next it is easy to compute that the set of lengths of $\hat{v}$-trajectories in $\hat{G}$ is $\{d : d|r\}$. Hence with **I**.7. and Proposition III.3., we conclude that the essential range of the multiplicity function of T_φ equals $\{d : d|r\}$.

And we see that if for $1 \le s \le m$, Υ_s denotes the $\hat{v}$-trajectory of the character dual to $e_1 + \ldots + e_s$, then $\#(\mathcal{H}_0 \cap \Upsilon_s) = s$. Therefore, with Propositions III.3. and III.4., we have obtained:

Proposition III.5. *The system* $(X \times G, T_\varphi, \mu \otimes m_G)$ *satisfies* $m(T_\varphi) = r$, *and the essential range of its multiplicity function is* $\{d : d|r\}$.

For $2 \le m < r$, *the system* $(X \times H, T_{\varphi_H}, \mu \otimes m_H)$ *satisfies* $m(T_{\varphi_H}) = m$, *and the essential range of its multiplicity function is* $\{1, \ldots, m\}$.

Now we pass to proving that $r(T_\varphi) = r$. Since $r(T_\varphi) \ge m(T_\varphi) = r$, we only prove that $r(T_\varphi) \le r$. Let us define

$$E_i := E_{t,i} = B_{t-1} \times \beta_{t,i},\ 1 \le i \le r.$$

We shall show that $\lim_{t\to\infty} t_\delta(E_1, \ldots, E_r, \omega) = 1$ for any $\delta > 0$.

For $0 \le u < l^r$, we have

$$(5)\qquad \begin{cases} (a) & u = u_1 + u_2 l + \ldots + u_r l^{r-1},\ 0 \le u_i < l,\ 1 \le i \le r, \\ (b) & \beta_i[u] = v^{i-1}(\beta_1[u]) = v^{i-1}(u_1 e_1 + \ldots + u_r e_r),\ 1 \le i \le r. \end{cases}$$

Let

$$\mathcal{U} = \{u = u_1 + u_2 l + \ldots + u_r l^{r-1} : 0 \le u_i < l,\ 2 \le i \le r,\ 0 \le u_1 < l - n\}.$$

Then $\#\mathcal{U} = l^{r-1}(l-n)$. For $u \in \mathcal{U}$, and $g = (g_1, \ldots, g_r) \in G$, for fixed $t \ge 0$, we have

$$\beta_1(g)[u] = (u_1 + g_1, \ldots, u_r + g_r) = \beta_1[u + u(g)],$$

where $u(g) = g_1 + g_2 l + \ldots + g_r l^{r-1}$, and $u(g) \le 2nl^{r-1}$. Hence $\frac{u(g)}{|\beta_1|} \le \frac{2n}{l}$ and

$$\bar{d}(\beta_{t,1}(g)[0, l_t^r - u(g) - 1], \beta_{t,1}[u(g), l_t^r - 1]) \le \frac{2n}{l_t} \to_t 0.$$

In a similar way we obtain, using $\beta_i = v^{i-1}(\beta_1)$, $1 \le i \le r$, and $(1b), (5b)$, that the above inequality is valid for $\beta_{t,i}$, $1 \le i \le r$.

Since $\omega_t = b_t \times b_{t+1} \times \ldots$ is a concatenation of the blocks $\beta_{t,i}(g)$, $g \in G$, $1 \le i \le r$, using **I**.2., **I**.4., and $(1c)$, we see that for given $\delta > 0$,

$$t_\delta(\{E_{t,i} : 1 \le i \le r\}, \omega) \ge 1 - \frac{2n}{l_t} \to_{t \to \infty} 1,$$

if $\frac{2n}{l_t} < \delta$, because as is obvious from **I**.4., if $0 \le \delta' < \delta$, then $t_\delta(\{E_{t,i} : 1 \le i \le r\}, \omega) \ge t_{\delta'}(\{E_{t,i} : 1 \le i \le r\}, \omega)$. Hence we obtain, using **I**.4. and Proposition III.5.:

Theorem (r, r)**.** *The system $(X \times G, T_\varphi, \mu \otimes m_G)$ satisfies $r(T_\varphi) = r = m(T_\varphi)$, and the essential range of its multiplicity function is $\{d : d|r\}$. It is measure theoretically isomorphic to a strictly ergodic continuous Morse automorphism.*

IV. The rank of T_{φ_H}: Theorem (m, r).

Using Theorem (r, r) and the natural factor P_H, it is obvious that $r(T_{\varphi_H}) \le r$. So from now on, we make some computations to show that $F^\star(\omega) < \frac{1}{r-1}$, because $F^\star(T_{\varphi_H}) \cdot r(T_{\varphi_H}) \ge 1$. We first of all proceed by introducing new blocks (over the alphabet H), more convenient than those having the subscript "H".

Recall that

$$\begin{cases} \omega_H = b_{0_H} \times b_{1_H} \times \ldots, \\ b_{t_H} = \beta_{t_H}^{p_t}, \\ \beta_{t_H} = \beta_{t,1_H} \ldots \beta_{t,r_H}, \\ \beta_{t,i_H} = F_{t,i_H} \times \ldots \times F_{t,r_H} \times F_{t,1_H} \times \ldots \times F_{t,i-1_H}. \end{cases}$$

Identifying H with $\mathbb{Z}_n^m = \tilde{e}_1 \mathbb{Z}_n + \ldots + \tilde{e}_m \mathbb{Z}_n$, where $\tilde{e}_1, \ldots, \tilde{e}_m$ are the natural generators of $\mathbb{Z}_n^m$, we get $\pi_H(e_i) = \tilde{e}_i$ for $1 \le i \le m$ and $\pi_H(e_i) = 0$ for $m+1 \le i \le r$. Then

$$\begin{cases} F_{t,i_H} = F_{i_H} = 0\tilde{e}_i(2\tilde{e}_i)\ldots((l-1)\tilde{e}_i) \text{ if } 1 \le i \le m, \\ F_{t,i_H} = F_{i_H} = \underbrace{0\ldots\ldots0}_{l_t \text{ times}} \text{ if } m+1 \le i \le r. \end{cases}$$

So from now on, we shall write $e_1, \ldots, e_m$, $F_i = F_{t,i}$, $\beta_i = \beta_{t,i}$, b_t, ω instead of $\tilde{e}_1, \ldots, \tilde{e}_m$, F_{t,i_H}, β_{t,i_H}, b_{t_H}, and ω_H respectively.

Therefore we shall make the forthcoming computations with the following new blocks (over H):

$$\begin{cases} F_{t,i} = F_i = 0e_i(2e_i)\dots((l-1)e_i) \text{ if } 1 \le i \le m, \\ F_{t,0} = F_0 = F_{t,i} = F_i = \underbrace{0\dots\dots0}_{\iota \text{ times}} \text{ if } m+1 \le i \le r, \end{cases}$$

and

$$\begin{cases} \beta_1 = F_1 \times \dots \times F_m \times \overbrace{F_0 \times \dots \times F_0}^{r-m}, \\ \beta_2 = F_2 \times \dots \times F_m \times \overbrace{F_0 \times \dots \times F_0}^{r-m} \times F_1, \\ \vdots \qquad \vdots \qquad \vdots \\ \beta_m = F_m \times \overbrace{F_0 \times \dots \times F_0}^{r-m} \times F_1 \times \dots \times F_{m-1}, \\ \beta_{m+1} = \overbrace{F_0 \times \dots \times F_0}^{r-m} \times F_1 \times \dots \times F_m, \\ \vdots \qquad \vdots \qquad \vdots \\ \beta_r = F_0 \times F_1 \times \dots \times F_m \times \overbrace{F_0 \times \dots \times F_0}^{r-m-1}. \end{cases}$$

Then $\beta = \beta_1 \dots \beta_r$, $b_t = \beta^{p_t}$, $\omega = b_0 \times b_1 \times \dots$ and $\omega_t = b_t \times b_{t+1} \times \dots$. Notice that for $t \ge 0$, $\omega = B_t \times \omega_{t+1}$.

In the sequel we shall need the formulas for the values $\beta_i[u]$, $0 \le u < l^r$. It follows from the definition of the F's and β's that ((5a))

$$\beta_i[u] = u_1^{(i)} e_1 + \dots + u_m^{(i)} e_m, \tag{6}$$

where

$$\begin{cases} (u_1^{(i)}, \dots, u_m^{(i)}) = (u_{r-i+2}, \dots, u_r, u_1, \dots, u_{m-i+1}) \text{ if } 1 \le i \le m, \\ (u_1^{(i)}, \dots, u_m^{(i)}) = (u_{r-i+2}, \dots, \dots, u_{r+m-i+1}) \text{ if } m+1 \le i \le r. \end{cases} \tag{7}$$

IV.1. Auxiliary Lemmas.

In this section we give some estimations of the $\bar{d}$-distance between blocks occurring in ω or ω_t. We assume that

$$l_t > \max\{16, \frac{8r}{m}\}, \ \textit{for every } t \ge 0. \tag{8}$$

Let

$$0 \le s' \le rl^r - 1, \ s' = i'l^r + u', \text{ with } 0 \le i' \le r-1, \ 0 \le u' \le l^r - 1, \tag{9}$$

and denote

$$I_{g,h} = \beta(g)\beta(h)[s', s' + rl^r - 1], \ g, h \in H.$$

Lemma IV.1. *If $1 \le i' \le r-2$ then for every $a, g, h \in H$,*

$$\bar{d}(\beta(a), I_{g,h}) \ge \frac{m}{6r}.$$

Proof. Using (3) we get

$$\bar{d}(\beta(a), I_{g,h}) \ge \frac{1}{3}\bar{d}(\check{\beta(a)}, \check{I_{g,h}}). \tag{10}$$

Take $0 \le s \le rl^r - 1$ and represent it as $s = il^r + u$ with $0 \le i \le r-1$ and $0 \le u \le l^r - 1$. Then

$$\check{\beta(a)}[s] = \check{\beta_i(a)}[u] \ if \ u > 0, \tag{11}$$

and

$$\check{\beta(a)}[s] = e = -(e_1 + \ldots + e_m) \ if \ u = 0. \tag{12}$$

Next let $\mathcal{S} = \{0 \le s \le rl^r - 1 : 0 \le u_1 \le l-2, \ where\ u\ is\ of\ the\ form\ (5a)\}$, and

$$\begin{cases} II &= \underbrace{e_1 \ldots e_1}_{l^r-1} e \underbrace{e_2 \ldots e_2}_{l^r-1} e \ldots e \underbrace{e_m \ldots e_m}_{l^r-1} e \underbrace{0 \ldots 0}_{l^r-1} e \ldots e \underbrace{0 \ldots 0}_{l^r-1}, \\ III &= IIeII[s', s' + rl^r - 2], \qquad\qquad (|II| = rl^r - 1). \end{cases}$$

It follows from (6) and (7) that whenever $u_1 < l-1$,

$$\begin{cases} \check{\beta}_i[u] = e_i \text{ if } 1 \le i \le m, \\ \check{\beta}_i[u] = 0 \text{ for } m+1 \le i \le r. \end{cases} \tag{13}$$

Using (11), (12) and (13) we obtain $\check{\beta(a)}[s] = II[s]$ if $s \in \mathcal{S}$. At the same time we have $\frac{\#\mathcal{S}}{rl^r - 1} \ge 1 - \frac{2}{l}$, which implies

$$\bar{d}(\check{\beta(a)}, II) < \frac{2}{l_t}. \tag{14}$$

Similarly we establish that

$$\bar{d}(\check{I_{g,h}}, III) < \frac{2}{l_t}. \tag{15}$$

It is easy to remark that

$$\bar{d}(II, III) \ge \frac{m}{r} \text{ (because } 1 \le i' \le r-2). \tag{16}$$

Now, using $(1g)$, $(14-16)$ we get
$\frac{m}{r} \le \bar{d}(II, III) \le \bar{d}(II, \check{\beta(a)}) + \bar{d}(\check{\beta(a)}, \check{I_{g,h}}) + \bar{d}(\check{I_{g,h}}, III) < \frac{4}{l_t} + \bar{d}(\check{\beta(a)}, \check{I_{g,h}})$.

The above, (8) and (10) imply $\bar{d}(\beta(a), I_{g,h}) \ge \frac{1}{3}(\frac{m}{r} - \frac{4}{l_t}) > \frac{m}{6r}$. ∎

Lemma IV.2. *Assume that*

$$\bar{d}(\beta(a'), I_{g',h'}[s', s'+rl^r-1]) < \delta, \text{ where } 0 < \delta < \min\{\frac{m}{6r}, \frac{1}{2r}, \frac{1}{16}\}, \tag{17}$$

and s' is as in (9), $a', g', h' \in G$, $u' = u'_1 + u'_2 l + \ldots + u'_r l^{r-1}$. Then either $i' = 0$ or $i' = r-1$, and

$$u'_1 \equiv \ldots \equiv u'_r \equiv p \mod n \text{ for some } p \in \mathbb{Z}_n. \tag{18}$$

Additionally,

$$\bar{d}(\beta(a), I_{g,h}[s', s'+rl^r-1]) \geq \frac{m}{8r}\nu, \tag{19}$$

for any $a, g, h \in G$, where $\nu = \frac{u'}{l^r}$ if $i' = 0$ and $\nu = 1 - \frac{u'}{l^r}$ if $i' = r-1$. Furthermore,
Case 1°: *if $i' = 0$ then*

$$a' - g' = p(e_1 + \ldots + e_m) \text{ and } u' < 4l^r(\delta + \frac{1}{l}). \tag{20}$$

If $g - a \neq g' - a'$ then

$$\bar{d}(\beta(a), I_{g,h}[s', s'+rl^r-1]) \geq (1-\nu) + \frac{m}{8r}\nu. \tag{21}$$

Case 2°: *if $i' = r-1$ then $a' - h' = p(e_1 + \ldots + e_m)$ and $u' > l^r(1 - 4\delta - \frac{1}{l})$. If $h - a \neq h' - a'$ then (21) holds.*

Proof. The inequality $\delta < \frac{m}{6r}$, (17) and Lemma IV.1. imply $i' = 0$ or $i' = r-1$. Assume that $i' = 0$. Using (10), (14), (15) (with a', g', h') we have

$$\bar{d}(\beta(a'), I_{g',h'}[s', s'+rl^r-1]) \geq \frac{1}{3}(\bar{d}(II, III) - \frac{4}{l}). \tag{22}$$

It is easy to see that $i' = 0$ implies $\bar{d}(II, III) \geq \frac{u'-1}{l^r}$. Thus (22) gives $\delta > \frac{1}{3}(\frac{u'-1}{l^r} - \frac{4}{l})$ which in turn implies that $u' < 4l^r(\delta + \frac{1}{l})$. This proves the second part of (20).

To prove the remaining part of the Lemma let us remark that the last inequality implies $u' < \frac{1}{2}l^r$ because $\delta < \frac{1}{4}$ and $\frac{4}{l_t} < \frac{1}{4}$ ((8)). Then using ($1f$) we get

$$\begin{aligned}&\bar{d}(\beta_i(a')[0, l^r - u' - 1], \beta_i(g')[u', l^r - 1]) \\ &\qquad \leq \frac{rl^r}{l^r - u'}\bar{d}(\beta(a'), I_{g',h'}[s', s'+rl^r-1]) < 2r\delta < 1,\end{aligned}$$

for every $1 \leq i \leq r$. Thus there exists at least one u, $0 \leq u \leq l^r - u' - 1$ (depending on i) such that $\beta_i[u] + a' = \beta_i[u + u'] + g'$. This equality and (6) (for u and u') give

$$u_1'^{(i)} e_1 + \ldots + u_m'^{(i)} e_m = a' - g'. \tag{23}$$

Then using (7) we obtain (18), which with (23) implies that $a_0 - g_0 = p(e_1 + \ldots + e_m)$. This completes the proof of (20).

Now we will prove (19) and (21). Using the same arguments as before we get that the equality $\beta_i[u]+a=\beta_i[u+u']$ for every $0\le u\le l^r-u'-1$ is equivalent to the condition ${u'_1}^{(i)}e_1+\ldots+{u'_m}^{(i)}e_m=a-g$. This equivalence and (23) then imply

$$\bar{d}(III_i,IV_i)=\begin{cases}0\ if\ g-a=g'-a',\\1\ if\ g-a\neq g'-a',\end{cases}\tag{24}$$

where $III_i=\beta_i(a)[0,l^r-u'-1]$, $IV_i=\beta_i(g)[u',l^r-1]$, $1\le i\le r$. Next, we need the following evident property;

$$\begin{cases}\text{for arbitrary subblocks } A_1\lessdot\beta_i(g'),\ A_2\lessdot\beta_{i'}(h'),\ i'\neq i,\ 1\le i\le m,\\ 1\le i'\le r,\ h',g'\in H,\text{ such that } |A_1|=|A_2|\ge 4,\text{ we have } \bar{d}(A_1,A_2)\ge\frac{1}{8}.\end{cases}\tag{25}$$

Let us examine the number u'. If $p>0$ ((18)) then $u'\ge l^{r-1}\ge 4$. If $p=0$ and $u'>0$ then $n|u'_i$ for $1\le i\le r$ and again $u'\ge n\ge 4$. Denoting

$$V_i=\beta_i(a)[l^r-u',l^r-1],\ VI_i=\beta_{i+1}(g)[0,u'-1],\ 1\le i\le m,$$

we have

$$|V_i|=|VI_i|=u'\ge 4.$$

It follows from (25) that

$$\bar{d}(V_i,VI_i)\ge\frac{1}{8}\ for\ 1\le i\le m.\tag{26}$$

Using $(1d)$, $(1e)$ and (26) we get
$\bar{d}(\beta(a),I_{g,h}[s',s'+rl^r-1])\ge$
$$\frac{1}{r}\left[(1-\tfrac{u'}{l^r})\textstyle\sum_{i=1}^r\bar{d}(III_i,IV_i)+\tfrac{u'}{l^r}\sum_{i=1}^m\bar{d}(V_i,VI_i)\right]\ge\tfrac{m}{8r}\tfrac{u'}{l^r}.$$

If in addition $g-a\neq g'-a'$ then (24) and (26) imply (21). This proves **Case 1°**. The proof of **Case 2°** is similar. ■

Lemma IV.3. *Let $s'_1=q'rl^r+i'l^r+u'$, $l=l_t$, $0\le q'\le p_t-1$, $0\le i'\le r-1$, $0\le u'\le l^r-1$, and let*

$$\widetilde{I_{g',h'}}=b_t(g')b_t(h')[s'_1,s'_1+p_trl^r-1].\tag{27}$$

Assume that $\bar{d}(b_t(a'),\widetilde{I_{g',h'}})<\delta$, where $0<\delta<\frac{1}{3}\min\{\frac{m}{6r},\frac{1}{2r},\frac{1}{16}\}$. Then $u'_1\equiv\ldots\equiv u'_r\equiv p\mod n$ for some $p\in\mathbb{Z}_n$ and either $i'=0$ or $i'=r-1$. If $s'_1\le\frac{1}{2}p_trl^r$ then $a'-g'=h_p$; and if $s'_1\ge\frac{1}{2}p_trl^r$ then $a'-h'=h_p$, where $h_p=p(e_1+\ldots+e_m)$.

Moreover, putting $\nu=\frac{u'}{l^r}$ if $i'=0$ and $\nu=1-\frac{u'}{l^r}$ if $i'=r-1$, $\zeta=\frac{q'}{p_t}$ if $s'_1\le\frac{1}{2}p_trl^r$ and $\zeta=1-\frac{q'}{p_t}$ if $s'_1\ge\frac{1}{2}p_trl^r$, we have:

Case 1°: *if $i'=0$ then $u'<l^r(4\delta+\frac{1}{l})$ and*

$$\bar{d}(b_t(a),\widetilde{I_{g,h}})\ge\tfrac{m}{8r}\nu\ for\ a,g,h\in H,\tag{28}$$
$$\bar{d}(b_t(a),\widetilde{I_{g,h}})\ge\tfrac{m}{8r}\nu+(1-\nu)(1-\zeta)\ if\ g-a\neq h_p\ and\ h-a=h_p,\tag{29}$$
$$\bar{d}(b_t(a),\widetilde{I_{g,h}})\ge\tfrac{m}{8r}\nu+(1-\nu)\zeta\ if\ g-a=h_p\ and\ h-a\neq h_p,\tag{30}$$
$$\bar{d}(b_t(a),\widetilde{I_{g,h}})\ge\tfrac{m}{8r}\nu+(1-\nu)\ if\ g-a\neq h_p\ and\ h-a\neq h_p.\tag{31}$$

Case 2°: *if* $i' = r-1$ *then* $u' > l^r(1-4\delta-\frac{1}{l})$ *and* $(28-31)$ *hold.*

Proof. Using $(1d)$ and $(1e)$ we obtain

$$\begin{aligned}\bar{d}(b_t(a'),\widetilde{I_{g',h'}}) &= (1-\tfrac{q'-1}{p_t})\bar{d}(\beta(a'),I_{g',g'}[s',s'+rl^r-1]) \\ &+ \tfrac{q'}{p_t}\bar{d}(\beta(a'),I_{h',h'}[s',s'+rl^r-1]) \\ &+ \tfrac{1}{p_t}\bar{d}(\beta(a'),I_{g',h'}[s',s'+rl^r-1]),\end{aligned} \tag{32}$$

where $s' = i'l^r + u'$. Assume that $s'_1 \le \frac{1}{2}p_t r l^r$. Then $1-\frac{q'-1}{p_t} \ge \frac{1}{3}$ and using (27), (32), we obtain the inequality $\bar{d}(\beta(a'),I_{g',g'}[s',s'+rl^r-1]) < 3\delta$. Then Lemma IV.1. implies $i'=0$ or $i'=r-1$. Now, we apply Lemma IV.2. with a',g' and $h' := g'$. We then have $u'_1 \equiv \ldots \equiv u'_r \equiv p \mod n$ and $a'-g' = h_p$. If $i'=0$ then $u' < l^r(4\delta+\frac{1}{l})$. Further (19) and (32) imply (28). Similarly (29), (30) and (31) are consequences of (19), (21) and (32).

If $i' = r-1$ then $u' > l^r(1-4\delta-\frac{1}{l})$ and the proof of the remaining part of the Lemma is the same. ■

Lemma IV.4. *Let* C,D *be blocks over* H *such that* $|C| = |D|+1$, $|D| = k \ge 1$. *Let* $0 \le s' \le p_t r l^r - 1$, $l = l_t$ *and*

$$I = b_t \times D, \quad II = (b_t \times C)[s', s'+kp_t r l^r - 1]).$$

Assume that $\bar{d}(I,II) < \delta$, *where* $0 < \delta < \frac{1}{3}\min\{\frac{m}{16r},\frac{1}{2r},\frac{1}{16}\}$. *Then there exists* $p \in \mathbb{Z}_n$ *such that*

$$\begin{cases}\bar{d}(D(h_p),C[0,k-1]) < \delta, \\ \text{or} \\ \bar{d}(D(h_p),C[1,k]) < \delta,\end{cases} \tag{33}$$

where

$$h_p = p(e_1+\ldots+e_m). \tag{34}$$

Proof. We let $I_j = b_t(D[j])$, $II_j = b_t(C[j])b_t(C[j+1])[s',s'+p_t r l^r-1]$, for $0 \le j \le k-1$. Then applying $(1d)$ we deduce that

$$\bar{d}(I,II) = \frac{1}{k}\sum_{j=0}^{k-1}\bar{d}(I_j,II_j) < \delta, \tag{35}$$

so there exists a $0 \le j_0 \le k-1$ such that $\bar{d}(I_{j_0},II_{j_0}) < \delta$. Let us suppose that $s' \le \frac{1}{2}p_t r l^r$. Then Lemma IV.3. implies that $D[j_0]-C[j_0] = h_p$ for some $p \in \mathbb{Z}_n$. Let

$$\begin{cases}\mathcal{Z}_0 = \{0 \le j \le k-1 : D[j]-C[j] \ne h_p\}, \\ \mathcal{Z}_1 = \{0 \le j \le k-1 : D[j]-C[j+1] \ne h_p\}.\end{cases}$$

Then using (35), defining for shortness $\bar{d}_j := \bar{d}(I_j,II_j)$, $0 \le j \le k-1$, we obtain

$$\delta > \bar{d}(I,II) = \frac{1}{k}\left(\sum_{j\in\mathcal{Z}_0\cap\mathcal{Z}_1}\bar{d}_j + \sum_{j\in\mathcal{Z}_0\setminus\mathcal{Z}_1}\bar{d}_j + \sum_{j\in\mathcal{Z}_1\setminus\mathcal{Z}_0}\bar{d}_j + \sum_{j\notin\mathcal{Z}_0\cup\mathcal{Z}_1}\bar{d}_j\right). \tag{36}$$

Using $(28-31)$ we get

$$\bar{d}(I_j, II_j)) \geq \frac{m}{8r}\nu + (1-\nu) \text{ if } j \in \mathcal{Z}_0 \cap \mathcal{Z}_1,$$
$$\bar{d}(I_j, II_j)) \geq \frac{m}{8r}\nu + (1-\nu)\zeta \text{ if } j \in \mathcal{Z}_1 \setminus \mathcal{Z}_0,$$
$$\bar{d}(I_j, II_j)) \geq \frac{m}{8r}\nu + (1-\nu)(1-\zeta) \text{ if } j \in \mathcal{Z}_0 \setminus \mathcal{Z}_1,$$
$$\bar{d}(I_j, II_j)) \geq \frac{m}{8r}\nu \text{ if } j \notin \mathcal{Z}_0 \cup \mathcal{Z}_1.$$

So (36) and the preceding four inequalities imply

$$\left\{\begin{array}{lll} \delta & > & \frac{m}{8r}\nu + \frac{1-\nu}{k}\left((1-\zeta)\#\mathcal{Z}_0 + \zeta\#\mathcal{Z}_1\right) \\ & \geq & \nu\left(\frac{m}{8r} - \frac{\min\{\#\mathcal{Z}_0, \#\mathcal{Z}_1\}}{k}\right) + \frac{\min\{\#\mathcal{Z}_0, \#\mathcal{Z}_1\}}{k}. \end{array}\right. \tag{37}$$

It follows from the above calculations that $\delta > \frac{1-\nu}{k}\min\{\#\mathcal{Z}_0, \#\mathcal{Z}_1\}$so since $\delta < \frac{m}{16r}$, we obtain $\frac{m}{8r} - \frac{\min\{\#\mathcal{Z}_0, \#\mathcal{Z}_1\}}{k} > 0$.

Thus with (37) we deduce that $\frac{\min\{\#\mathcal{Z}_0, \#\mathcal{Z}_1\}}{k} < \delta$. Finally let us observe that $\frac{\min\{\#\mathcal{Z}_0, \#\mathcal{Z}_1\}}{k} = \min\{\bar{d}(D(h_p), C[0,k-1]), \bar{d}(D(h_p), C[1,k])\}$, from which (33) follows. ■

Corollary IV.1. *Under the assumptions of Lemma IV.4., if additionally we assume that* $4\delta k < 1$, *then* $D(h_p) = C[0, k-1]$ *if* $\zeta \leq \frac{1}{2}$ *and* $D(h_p) = C[1,k]$ *if* $\zeta > \frac{1}{2}$.

Proof. If follows from the definition of ν (Lemma IV.2.) that $1-\nu \geq \frac{1}{2}$ if $4\delta + \frac{1}{l_t} < \frac{1}{2}$ (use also (8)). Then (37) implies $\delta > \frac{\#\mathcal{Z}_0}{4k}$ if $\zeta \leq \frac{1}{2}$ and $\delta > \frac{\#\mathcal{Z}_1}{4k}$ if $\zeta > \frac{1}{2}$. Thus either $\#\mathcal{Z}_0 = 0$ or $\#\mathcal{Z}_1 = 0$. ■

Lemma IV.5. *Let* C, D *be blocks over* H *such that* $k = |D| \geq 1$ *and* $|C| = |D|+1$. *Let* $0 \leq \tilde{s} \leq n_t - 1$ *and* $\tilde{I} = B_t \times D$, $\tilde{II} = (B_t \times C)[\tilde{s}, \tilde{s} + kn_t - 1])$. *Suppose that*

$$\bar{d}(\tilde{I}, \tilde{II}) < \delta \text{ where } 0 < \delta < \frac{1}{3}\min\{\frac{m}{16r}, \frac{1}{2r}, \frac{1}{16}\}. \tag{38}$$

Then there exists $p \in \mathbb{Z}_n$ *such that*

$$\left\{\begin{array}{l} \bar{d}(D(h_p), C[0,k-1]) < \delta, \\ \text{or} \\ \bar{d}(D(h_p), C[1,k]) < \delta, \end{array}\right. \tag{39}$$

where h_p *is defined by* (34).

Proof. We use an induction argument on t. For $t = 0$ the Lemma follows from Lemma IV.4.. So let us suppose that (39) is true for $t-1$ and assume that (38) holds for t.

We have $B_t \times C = B_{t-1} \times (b_t \times C)$, $B_t \times D = B_{t-1} \times (b_t \times D)$, and $\tilde{s} = \tilde{s}_1 n_{t-1} + \tilde{s}_2$, where $0 \leq \tilde{s}_1 \leq \lambda_t - 1$, and $0 \leq \tilde{s}_2 \leq n_{t-1} - 1$.

So let $D_1 = b_t \times D$ and $C_1 = b_t \times C[\tilde{s}_1, \tilde{s}_1 + \lambda_t(k+1) - 1]$. Then (38) can be rewritten as $\bar{d}(B_{t-1} \times D_1, (B_{t-1} \times C_1)[\tilde{s}_2, \tilde{s}_2 + kn_t - 1]) < \delta$. Using the induction hypothesis we obtain

$$\bar{d}(D_1(h_{p_1}), C_1[0, k\lambda_t - 1]) < \delta, \tag{40}$$

or

$$\bar{d}(D_1(h_{p_1}), C_1[1, k\lambda_t]) < \delta, \tag{41}$$

for some $p_1 \in \mathbb{Z}_n$. Let us suppose for instance that (40) is holding. Then it can be rewritten as $\bar{d}((b_t \times D)(h_{p_1}), (b_t \times C)[\tilde{s}_1, \tilde{s}_1 + k\lambda_t - 1]) < \delta$. Then Lemma IV.4. implies that either $\bar{d}(D((h_{p_1}))(h_{p_2}), C[0,k-1]) < \delta$ or $\bar{d}((D(h_{p_1}))(h_{p_2}), C[1,k]) < \delta$ for some $p_2 \in \mathbb{Z}_n$. Letting $p = p_1 + p_2$, these two last eventualities read as (39). ■

IV.2. Special subsequences and subblocks of ω.

To estimate the numbers $t_\delta(A,\omega)$ for the blocks A appearing in ω we distinguish special subsequences of ω and then we examine the possible appearances of all long enough blocks in those subsequences.

Every fragment $I_j := \omega[jn_t, (j+1)n_t - 1]$, $j \geq 0$, is of the form $B_t(h)$ for some $h \in H$. So by a *t-symbol* of ω we mean a fragment of ω like I_j, that corresponds to a block $B_t(h)$.

Given a fragment $A = \omega[q, q+s-1]$ of ω we define

$$A^\delta = \omega[q - \delta s, q + s + \delta s - 1],$$

where $0 < \delta < \frac{1}{2}$.

Gathering the t-symbols $B_t(h)$ from their natural positions (like for I_j) we can define disjoint subsequences $\omega_t(h)$ of ω, $h \in H$. Precisely, let $\mathbb{N}_h = \{j \geq 0 : I_j = B_t(h)\}$ and put $\omega_t(h) = \cup_{j \in \mathbb{N}_h} I_j$. In the same way we define

$$\omega_t^\delta(h) = \cup_{j \in \mathbb{N}_h} I_j^\delta. \tag{41a}$$

Other natural subblocks of ω that we need to distinguish are the blocks

$$E_{t,i}(h) = B_t \times \beta_{t+1,i}(h),\ 1 \leq i \leq r,\ h \in H.$$

Then every fragment $II_j = \omega[jn_t l_{t+1}^r, (j+1)n_t l_{t+1}^r - 1]$, $j \geq 0$, is equal to some $E_{t,i}(h)$, for some $1 \leq i \leq r,\ h \in H$. Then we define additional subsequences of ω as follows:

$$\left\{\begin{array}{l} \mathbb{N}_i(h) = \{j \geq 0 : II_j = E_{t,i}(h)\}, \\ \omega_{t+1,i}(h) = \cup_{j \in \mathbb{N}_i(h)} II_j, \\ \omega_{t+1,i}^\delta(h) = \cup_{j \in \mathbb{N}_i(h)} II_j^\delta. \end{array}\right. \tag{41b}$$

The blocks

$$L_{t,i,g} := \overbrace{g \ldots\ldots g}^{l^{r-i+1} \text{ times}},\ l = l_{t+1},\ i = m+1, \ldots, r,\ g \in H,$$

also appear naturally in ω_{t+1}. Any block $\beta_{t+1,i}(h)$ is a concatenation of the blocks $L_{t,i,g}$, where g runs over H; moreover,

$$fr(L_{t,i,g}, \beta_{t+1,i}(h)) = \frac{1}{n^m},\ g, h \in H,\ m+1 \leq i \leq r.$$

Define $M_{t,i,g} = B_t \times L_{t,i,g}$, $m+1 \leq i \leq r,\ g \in H$. The blocks $M_{t,i,g}$ appear in ω at the positions $jn_t l^r + sl^{r-i+1},\ l = l_{t+1},\ j \in \mathbb{N}_i(h),\ 0 \leq s < l^{i-1},\ h \in H$. Let us denote and define the following:

$$III_{j,s} = III_{t,j,s} = \omega[jn_t l^r + sl^{r-i+1}, jn_t l^r + (s+1)l^{r-i+1} - 1],$$
$$j \in \cup_{h \in H} \mathbb{N}_i(h),\ 0 \leq s < l^{i-1},$$

$$\left\{\begin{array}{ll} \mathbb{N}_{i,g} & = \{(j,s) : j \in \cup_{h \in H} \mathbb{N}_i(h),\ 0 \leq s < l^{i-1},\ III_{j,s} = M_{t,i,g}\}, \\ \omega_{t+1,i,g} & = \cup_{(j,s) \in \mathbb{N}_{i,g}} III_{j,s}, \\ \omega_{t+1,i,g}^\delta & = \cup_{(j,s) \in \mathbb{N}_{i,g}} III_{j,s}^\delta. \end{array}\right. \tag{41c}$$

The subsequences defined above ($(41a-c)$) enjoy the following properties (for fixed $t \geq 0$):

(42) $$\begin{cases} \omega_t(h) \text{ are pairwise disjoint when } h \text{ runs over } H, \cup_{h\in H}\omega_t(h) = \omega, \\ \text{and } D(\omega_t(h), \omega) = \frac{1}{n^m},\ h \in H, \end{cases}$$

(43) $$\begin{cases} \omega_{t+1,i}(h) \text{ are pairwise disjoint when } h \text{ runs over } H \text{ and } 1 \leq i \leq r, \\ \cup_{h\in H} \cup_{i=1}^{r} \omega_{t+1,i}(h) = \omega, \text{ and } D(\omega_{t+1,i}(h), \omega) = \frac{1}{rn^m}, \end{cases}$$

(44) $$\begin{cases} \omega_{t+1,i,g} \text{ are pairwise disjoint when } g \text{ runs over } H \text{ and } m+1 \leq i \leq r, \\ \cup_{g\in H}\omega_{t+1,i,g} = \cup_{h\in H}\omega_{t+1,i}(h), \text{ and } D(\omega_{t+1,i,g}, \omega) = \frac{1}{(r-m)n^m}. \end{cases}$$

Until the end of the paper, we shall assume that δ and t satisfy

(45) $$\delta n_t \geq 4 \quad \textit{and} \quad \delta l_{t+1} \geq 4.$$

Now, we classify the subblocks $A \lessdot \omega$ such that $|A| \geq 3n_0$. For every such block there exists a unique $t \geq 0$ such that

(46) $$A = E_1(B_t \times C)E_2,$$

where $|C| \geq 1$, $C \lessdot b_{t+1}(g')b_{t+1}(h')$ for some $g', h' \in H$ such that $(g')(h') \lessdot \omega_{t+2}$, E_1 (resp. E_2) is a right-hand side (resp. a left-hand side) of a t-symbol.

Given $\delta > 0$ and $t \geq 0$ satisfying (45), we define three subsequences $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3$ of ω by

$$\begin{bmatrix} \tilde{\omega}_{t,1} = \tilde{\omega}_1 = \cup_{j\geq 0}\omega[(j-\delta)n_t, (j+\delta)n_t], \\ \tilde{\omega}_{t,2} = \tilde{\omega}_2 = \cup_{j\geq 0}\omega[(j-\delta)l_{t+1}^r, (j+\delta)l_{t+1}^r], \\ \tilde{\omega}_{t,3} = \tilde{\omega}_3 = \cup_{j\geq 0}\omega[(j-\delta)l_{t+1}, (j+\delta)l_{t+1}]. \end{bmatrix}$$

Notice that $D(\tilde{\omega}_1 \cup \tilde{\omega}_2 \cup \tilde{\omega}_3, \omega) \leq 6\delta$, where $\tilde{\omega}_1 \cup \tilde{\omega}_2 \cup \tilde{\omega}_3$ denotes the obvious subsequence of ω. We shall examine blocks (46) satisfying the additional condition

(47) $$A \cap (\omega \setminus (\tilde{\omega}_1 \cup \tilde{\omega}_2 \cup \tilde{\omega}_3)) \neq \emptyset.$$

We now distinguish six classes $\mathcal{K}_1, \ldots, \mathcal{K}_6$ for blocks of the form (46) that are such that any A satisfying (47) belongs to at least one of these classes:

(A) $$A \in \mathcal{K}_1 \textbf{ iff } |C| \leq 3,$$

(B) $$A \in \mathcal{K}_2 \textbf{ iff } |C| \geq 4 \textit{ and } C \lessdot \beta_{t+1,i}^{\delta}(h') \textit{ where } h' \in H \textit{ and } 1 \leq i \leq m,$$

(C) $$\begin{cases} A \in \mathcal{K}_3 \textbf{ iff } C \lessdot \beta(g')\beta(g'),\ \beta := \beta_{t+1},\ \exists C_1C_2 \lessdot C, \\ \frac{|C_j|}{|\beta_i|} > \delta,\ j = 1, 2,\ C_1 \lessdot \beta_i(g'), \\ C_2 \lessdot \beta_{i+1}(g'),\ 1 \leq i \leq m,\ \textit{or } i = r \textit{ and } r+1 := 1,\ g' \in H, \end{cases}$$

(D) $$\begin{cases} A \in \mathcal{K}_4 \textbf{ iff } C \lessdot \beta(g')\beta(g'),\ \beta := \beta_{t+1},\ C \lessdot L_{t,i',g'}^{\delta} L_{t,i',h'}^{\delta}, \\ |C| \geq 4,\ m+1 \leq i' \leq r,\ g', h' \in H,\ g' \neq h', \end{cases}$$

(E) $$\begin{cases} A \in \mathcal{K}_5 \textbf{ iff } C \lessdot \beta(g)\beta(g),\ \beta := \beta_{t+1},\ \textit{for some } g \in H, \\ \textit{and there are at least three blocks } L_{t,i,g'},\ L_{t,i,h'},\ L_{t,i,f'} \\ \textit{such that } fr(L_{t,i,g'}, C) > \delta, fr(L_{t,i,h'}, C) > \delta, fr(L_{t,i,f'}, C) > \delta, \\ \textit{where } g', h', f' \textit{ are pairwise distinct}, \end{cases}$$

(F) $$A \in \mathcal{K}_6 \textbf{ iff } \beta_{t+1}(g') \lessdot C \textit{ for some } g' \in H.$$

IV.3. Theorem (m, r).

Let $\delta_0 = \min\{\frac{m}{8r}, \frac{1}{2r}, \frac{1}{16}\}$, A be as in (47) and $I = \omega[q, q + |A| - 1]$. We shall estimate $t_{\delta'}(A, \omega)$.

Lemma IV.6. *Assume that* $\bar{d}(A, I) < \delta$, $0 < \delta < \frac{1}{9}\delta_0$, $l_t \geq 4$ *and* $A \in \mathcal{K}_1$ *(cf. (A)). Then* $I \lessdot \cup_{h \in H_A} \omega_t^3(h)$, *where* $H_A \subset H$ *with* $\# H_A \leq n$. *Hence* $t_\delta(A, \omega) \leq \frac{7}{n^{m-1}} + 6\delta$.

Proof. Let $q_1 = |E_1|$ $(q_1 < n_t)$, and $k = |C|$. Using $(1f)$ we have $3\delta > 3\bar{d}(A, I) \geq \bar{d}(B_t \times C, \omega[q + q_1, q + q_1 + kn_t - 1])$. So applying Lemma IV.5. we get

$$\bar{d}(B_t \times C, \omega[sn_t, (s + k)n_t - 1]) < 3\delta, \tag{48}$$

where $|sn_t - q - q_1| \leq n_t$. This implies

$$|sn_t - q| < 2n_t. \tag{49}$$

Next, (48) gives $\bar{d}(C, \omega_{t+1}[s, s + k - 1]) < 3\delta$, and Corollary IV.1. (it holds that $4k\delta < 1$) implies that either $C(h_p) = \omega_{t+1}[s, s + k - 1]$ or $C(h_p) = \omega_{t+1}[s + 1, s + k]$ for some $p \in \mathbb{Z}_n$. Putting $p_1 = C[1] + h_p$ if $|C| = 3$ and $C[0] + h_p$ if $|C| \leq 2$, we get that either $\omega_{t+1}[s + 1] = p_1$ (if $k = 3$) or $\omega_{t+1}[s] = p_1$ (if $k \leq 2$).

Assume that $k = 3$ for instance. The equality $\omega_{t+1}[s + 1] = C[1] + h_p$ means that the fragment $\omega[(s + 1)n_t, (s + 2)n_t - 1] \lessdot \cup_{p \in \mathbb{Z}_n} \omega_t(C[1] + h_p)$. Taking into consideration two neighbouring t-symbols from the right and left sides, plus the blocks E_1 and E_2, we deduce that $I \lessdot \cup_{h \in H_A} \omega_t^3(h)$ where $\# H_A \leq n$.

In the case $k \leq 2$ we obtain the same conclusion. Using (42) and (47) we deduce that $t_\delta(A, \omega) \leq \frac{7n}{n^m} + 6\delta = \frac{7}{n^{m-1}} + 6\delta$. ■

Lemma IV.7. *Assume that* $\bar{d}(A, I) < \frac{\delta^2}{6}$, $0 < \delta < \frac{1}{9}\delta_0$, *and* $A \in \mathcal{K}_2$ *(see (B)). Then* $I \lessdot \cup_{h \in H} \omega_{t+1,i}^{(2\delta)}(h)$. *Hence* $t_{\frac{\delta^2}{6}}(A, \omega) \leq \frac{1+4\delta}{r} + 6\delta$.

Proof. The same arguments than those appearing in Lemma IV.6. lead to $\bar{d}(C, \omega_{t+1}[s, s + k - 1]) < \frac{\delta^2}{2}$. The conditions (47) and $\delta l_{t+1} \geq 4$ imply that C contains a subblock $\tilde{C}$ such that $\frac{|\tilde{C}|}{|C|} \geq \frac{1}{2}$ and $\tilde{C} \lessdot \beta_{t+1,i}(h')$ for some $h' \in H$. Hence $\bar{d}(\tilde{C}, \omega[s_1, s_1 + |\tilde{C}| - 1]) < \delta^2$, where $s_1, \ldots, s_1 + |\tilde{C}| - 1$ are the positions of $\tilde{C}$ in C.

Writing C and $\omega[s, s + k - 1]$ instead of $\tilde{C}$ and $\omega[s_1, s_1 + |\tilde{C}| - 1]$ respectively, suppose that $\omega_{t+1}[s, s + k - 1]$ contains a subblock D_1 such that $|D_1| \geq 4$, $\frac{|D_1|}{k} > \delta$, $D_1 \lessdot \cup_{h \in H} \omega_{t+1,i}(h)$ and $i \neq i'$. Denoting by C_1a subblock of C appearing at the same positions as D_1 in $\omega_{t+1}[s, s + k - 1]$ and using $(1f)$ we get
$\delta^2 > \bar{d}(C, \omega_{t+1}[s, s + k - 1]) \geq \frac{|D_1|}{k}\bar{d}(C_1, D_1) > \delta\bar{d}(C_1, D_1)$,
what gives $\bar{d}(C_1, D_1) < \delta$. However the property (25) says that $\bar{d}(C_1, D_1) \geq \frac{1}{8}$. Hence either $\frac{|D_1|}{k} < \delta$ or $|D_1| \leq 3$. In both cases this means that $B_t \times \omega_{t+1}[s, s + k - 1] \lessdot \cup_{h \in H} \omega_{t+1,i}^\delta(h)$ because $k \leq |\beta_i|$ and $\frac{3}{l_{t+1}^r} < \delta$ (cf. (45)). Enclosing the (wings) blocks E_1, E_2 we obtain $I \lessdot \cup_{h \in H} \omega_{t+1,i}^{2\delta}(h)$ on the base of (49) and the inequality $\frac{2}{n_t} < \delta$.

We conclude as in the above Lemma, using (47) and (43), to obtain $t_{\frac{\delta^2}{6}}(A, \omega) \leq \frac{1+4\delta}{r} + 6\delta$. ■

Lemma IV.8. *Let* $\bar{d}(A,I) < \frac{\delta^3}{36r}$, $0 < \delta < \frac{1}{9}\delta_0$, *and assume* $A \in \mathcal{K}_3$ *(see (C)). Then either*

(50) $I \lessdot \cup_{h \in H_A} \omega_{t+2}^{(2\delta)}(h)$, $\#H_A \le n$, *then* $t_{\frac{\delta^3}{36r}}(A,\omega) \le \frac{1+4\delta}{n^{m-1}} + 6\delta$,

or

(51) $I \lessdot \cup_{h \in H} \omega_{t+1,i'}^{(2\delta)}(h)$, $i' = i$ *or* $i' = i+1$, *then* $t_{\frac{\delta^3}{36r}}(A,\omega) \le \frac{1+4\delta}{r} + 6\delta$,

where i is defined in (C).

Proof. As before we have $\bar{d}(C, \omega_{t+1}[s, s+k-1]) < \frac{\delta^3}{12r}$. Let D_1 and D_2 be subblocks of $\omega_{t+1}[s, s+k-1]$ occupying the same positions in it as C_1 and C_2 do in C, respectively. Then we have $\frac{|C_j|}{|C|} = \frac{|C_j|}{|\beta_i|}\frac{|\beta_i|}{|C|} \ge \frac{\delta}{2r}$. Hence

(52) $$\bar{d}(C_1, D_1) < \frac{\delta^2}{6} \text{ and } \bar{d}(C_2, D_2) < \frac{\delta^2}{6}.$$

It follows from the proof of Lemma IV.7. that $D_1 \lessdot \cup_{h \in H} \omega_{t+1,i}^{(2\delta)}(h)$ and $D_2 \lessdot \cup_{h \in H} \omega_{t+1,i+1}^{(2\delta)}(h)$ (these last inclusions are also valid for $i = m$ and $i = r$). From the above inclusions we deduce that $D_1 D_2 \lessdot \beta_{t+1,i}^{(2\delta)}(g_1)\beta_{t+1,i+1}^{(2\delta)}(g_1)$ for some $g_1 \in H$. If in addition $D_1 \lessdot \beta_{t+1,i}^{(2\delta)}(g_1)$ (or $D_2 \lessdot \beta_{t+1,i+1}^{(2\delta)}(g_1)$) then (51) holds. If not then

(53) $$D_1 D_2 \lessdot \beta_{t+1,i}^{(2\delta)}(g_1)\beta_{t+1,i+1}^{(2\delta)}(g_1)$$

Further, $(1c, e)$ and (52) imply $\bar{d}(b_t \times C_1C_2, b_t \times D_1D_2) < \frac{\delta^2}{3}$. Applying Lemma IV.4. we deduce

(54) $\bar{d}((C_1C_2)(h_p), (D_1D_2)[0, |C_1C_2| - 1]) < \frac{\delta^2}{3}$, *or*

(55) $\bar{d}((C_1C_2)(h_p), (D_1D_2)[1, |C_1C_2|]) < \frac{\delta^2}{3}$, *for some* $p \in \mathbb{Z}_n$.

Suppose that (54) holds. Then we can write $C_1 = E(g_0)$, $C_2 = E'(g_0)$, $D_1 = E(g_1)$, $D_2 = E'(g_1)$, where E is a subblock of the right side of β_i and E' comes from the left side of β_{i+1}. Thus (54) has a form $\bar{d}((EE')(g_0 + h_p), (EE')(g_1)) < \frac{\delta^2}{3}$. Hence we must have $g_1 - g_0 = h_p$. Putting $H_A = \{g_0 + h_p : p \in \mathbb{Z}_n\}$, we deduce from (53) that (50) is satisfied. In the same way we deduce (50) from (55). If (50) holds then using (42) and (47) we have $t_{\frac{\delta^3}{36r}}(A,\omega) \le \frac{1+4\delta}{n^{m-1}} + 6\delta$. Else for (51) we use (43) instead of (42) and deduce $t_{\frac{\delta^3}{36r}}(A,\omega) \le \frac{1+4\delta}{r} + 6\delta$. ■

Lemma IV.9. *Let* $\bar{d}(A,I) < \frac{\delta}{2}$, $0 < \delta < \frac{1}{9}\delta_0$, *and* $A \in \mathcal{K}_4$ *(see (D)). Then*

(56) $I \lessdot \cup_{i=m+1}^{r} \omega_{t+1,i,g'}^{(1+2\delta)}$ *for some* $g' \in H$. *Hence* $t_{\frac{\delta}{2}}(A,\omega) \le \frac{3+4\delta}{n^m} + 6\delta$

Proof. The assumptions of the Lemma imply

(57) $$\bar{d}(C, \omega_{t+1}[s, s+k-1]) < \frac{\delta}{2}.$$

First assume that $C \lessdot L_{t,i_0,g'}$. Then (57) implies directly that $\omega_{t+1}[s, s+k-1] \lessdot \omega_{t+1,i_0,g'}$ for some $m+1 \le i_0 \le r$. Using the same arguments as in the preceding Lemmas we get $I \lessdot \cup_{i=m+1}^{r} \omega_{t+1,i,g'}^{(2\delta)}$.

Now assume that $C \lessdot L_{t,i_0,g'}^{\delta} L_{t,i_0,h'}^{\delta}$. The conditions (47) and the inequality $\delta l_{t+1} \ge 4$ imply that C contains a subblock C_1 such that $\frac{|C_1|}{|C|} \ge \frac{1}{2}$ and $C_1 \lessdot L_{t,i,g''}$, where $g' = g'$ or h'. Repeating the above arguments we obtain $I_1 = (B_t \times C_1) \lessdot \cup_{i=m+1}^{r} \omega_{t+1,i,g'}^{(2\delta)}$. This implies (56). Then (47) and (44) imply that $t_{\frac{\delta}{2}}(A,\omega) \le \frac{3+4\delta}{n^m} + 6\delta$. ■

Lemma IV.10. *Let* $\bar{d}(A,I) < \frac{\delta^2}{18},\ 0 < \delta < \frac{1}{9}\delta_0,\ A \in \mathcal{K}_5$ *(see (E)). Then*

(58) $$I \lessdot \cup_{h\in H_A}\omega_{t+2}^{(2\delta)}(h),\ \#H_A \le n.\ \text{Hence}\ t_{\frac{\delta^2}{18}}(A,\omega) \le \frac{1+4\delta}{n^{m-1}} + 6\delta.$$

Proof. As before we obtain

(59) $$\bar{d}(C, \omega_{t+1}[s, s+k-1]) < \frac{\delta^2}{6}.$$

Let us assume that C contains exactly three kinds of subblocks $L_{t,i,g'}, L_{t,i,h'}, L_{t,i,f'}$, each appearing in its natural positions. Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be the families of all subblocks $L_{t,i,g'}, L_{t,i,h'}, L_{t,i,f'}$ of C appearing at their natural positions. Each of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ is a union of disjoint subblocks of C.

We pick subblocks D_1, D_2, D_3 from $\omega_{t+1}[s, s+k-1]$ occupying in it the same positions as the blocks C_1, C_2, C_3 from the families $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ do in C, respectively. They define the families $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$. For each $C_j \in \mathcal{C}_j$, we let $D_j(C_j)$ denote the block of $\mathcal{D}_j$ appearing in the same positions as C_j, for $j = 1, 2, 3$. Then we define

$$\bar{d}(\mathcal{C}_j, \mathcal{D}_j) := \frac{1}{\#\mathcal{C}_j} \sum_{C_j \in \mathcal{C}_j} \bar{d}(C_j, D_j(C_j)).$$

Then (59) implies $\bar{d}(\mathcal{C}_j, \mathcal{D}_j) < \frac{\delta}{2},\ 1 \le j \le 3$. It is not hard to deduce that $I \lessdot \omega_{t+2}^{(2\delta)}(h)$ for some $h \in H$. Let g be another element of H such that $I \lessdot \omega_{t+2}^{(2\delta)}(g)$. Then using Lemma IV.4. and repeating the arguments of the proof of Lemma IV.8. we obtain $g - h = h_p$ for some $p \in \mathbb{Z}_n$. This implies (58). The same arguments apply to the case where C contains more than three blocks of the form $L_{t,i,g'}$.

Then using (42) and (47) we deduce that $t_{\frac{\delta^2}{18}}(A,\omega) \le \frac{1+4\delta}{n^{m-1}} + 6\delta$. ■

Lemma IV.11. *Let* $\bar{d}(A,I) < \frac{\delta^2}{18},\ 0 < \delta < \frac{1}{9}\delta_0$*, and* $A \in \mathcal{K}_6$ *(see (F)). Then*

(60) $$I \lessdot \cup_{h\in H_A}\omega_{t+2}^{(2\delta+2)}(h),\ \#H_A \le n.\ \text{Hence}\ t_{\frac{\delta^2}{18}}(A,\omega) \le \frac{5+4\delta}{n^{m-1}} + 6\delta.$$

Proof. Using the same arguments as in the (numerous) preceding Lemmas we prove that C contains a subblock C_1 such that $\frac{|C_1|}{|C|} \ge \frac{1}{2}$, and C_1 contains at least one block $\beta_{t+1}(g')$, and $C_1 \lessdot b_{t+1}(g')$ (or $C_1 \lessdot b_{t+1}(h')$). Then

(61) $$\bar{d}(C_1, D_1) < \frac{\delta}{3},$$

where D_1 is defined in the same way as the one in the proof of Lemmas IV.8., IV.10.. Then (61) implies that

(62) $$D_1 \lessdot b_{t+1}(g) \text{ for some } g \in H.$$

Then we use Lemma IV.4. to deduce that $g - g' = h_p$ for some $p \in \mathbb{Z}_n$. This, with (62), implies (60). Then (47) and (42) imply $t_{\frac{\delta^2}{18}}(A,\omega) \le \frac{5+4\delta}{n^{m-1}} + 6\delta$. ■

Theorem (m, r)**.** *The system* $(X \times H, T_{\varphi_H}, \mu \otimes m_H)$ *is such that* $r(T_{\varphi_H}) = r$*,* $m(T_{\varphi_H}) = m$*, and the essential range of its multiplicity function is* $\{1, \dots, m\}$*. It is measure theoretically isomorphic to a strictly ergodic continuous Morse automorphism.*

Proof. Let $0 < \delta_2 < \frac{1}{9}\delta_0$ and let $\delta_1 = \frac{\delta_2^3}{36r}$. Then select t_0 such that $\delta_2 n_{t_0} \ge 4$, $\delta_2 l_{t_0+1} \ge 4$. Let $A \lessdot \omega$ be a block such that $|A| \ge 3n_{t_0}$. Then A has a form (46) with $t \ge t_0$. With Lemmas IV.6.-IV.11. we deduce that, using the fact that for $0 \le \delta' < \delta,\ t_\delta(A,\omega) \ge t_{\delta'}(A,\omega)$,

$$t_{\delta_1}(A,\omega) \le \max\{\frac{7}{n^{m-1}}, \frac{1+4\delta_2}{r}, \frac{3+4\delta_2}{n^m}, \frac{1+4\delta_2}{n^{m-1}}, \frac{5+4\delta_2}{n^{m-1}}\} + 6\delta_2 := a.$$

Hence $F^\star(T_{\varphi_H}) \le a$. We can select n and δ_2 in order to ensure that $a < \frac{1}{r-1}$. Then using **I**.4. we deduce that $r(T_{\varphi_H}) \ge r$. ■

Remark IV.1. *A "Chacon type" modification of the candidate to the (m,r) pair (see* **II**.*,* **IV**.*) can lead to a weakly mixing system realizing the pair (m,r) ($2 \le m \le r < \infty$). Namely, we define the base system (X,T,μ) as follows (it shall no longer be an adic adding machine):*
1): *pick $\lambda_0, \lambda_1, \ldots$ as in* **II**.*;*
2): *define the generating partition of the continuous Lebesgue probability space (X,μ) inductively as to be a refining sequence $(\xi_t)_{t\ge 0}$ of T-towers*

$$\xi_t = (D_0^t, \ldots, D_{n_t-1}), \quad TD_i^t = D_{i+1}^t,\ 0 \le i \le n_t - 2,$$

where $n_0 = \lambda_0$, $\mu(D_i^0) = \frac{1}{n_0}$ ($0 \le i < n_0$), and if ξ_t is defined, then we let $n_{t+1} = \lambda_{t+1} n_t + 1$, and

$$\left(\cup_{j=0}^{\lambda_{t+1}-2} D_{jn_t+i}^{t+1}\right) \cup D_{n_{t+1}-n_t+i+1}^{t+1} = D_i^t,\ 0 \le i < n_t,$$

and $\mu(D_u^{t+1}) = \frac{1}{n_{t+1}}$, $0 \le u < n_{t+1}$.
3): *define the sequence of blocks $(b_t)_{t\ge 0}$ exactly as in* **II**. *if $m = r$ or* **IV**. *if $m < r$;*
4): *with the new sequence $(n_t)_{t\ge 0}$ from* **2)** above, define the blocks $(B_t)_{t\ge 0}$ by $B_0 = b_0$ and

$$B_{t+1} = B_t(b_{t+1}[0]) \ldots B_t(b_{t+1}([\lambda_{t+1}-2])0B_t(b_{t+1}[\lambda_{t+1}-1]);$$

5): *let the M-cocycle $\varphi : X \to H$ ($H = G$ if $m = r$) be defined by $\varphi_{|D_i^t} = B_t[i+1] - B_t[i]$, $0 \le i < n_t - 1$.*

Then $(n_t)_{t\ge 0}$ is a rigid time for T, and following the argumentation from [KwJLe, Sec. 4], $(X \times H, T_\varphi, \mu \otimes m_H)$ is seen to be weakly mixing with the desired spectral multiplicity equal to m. As is seen along the lines of [Kw] or [Le], the system has a strictly ergodic shift representation $(\Omega_\omega, S, \mu_\omega)$ where $\omega \in H^{\mathbb{N}}$ is defined by

$$\omega[0, n_t - 1] = B_t, \quad t \ge 0.$$

The computation of its rank uses this symbolic representation and may be done in a closely similar way to what was done in **III**. *and* **IV**..

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