Finite rank transformation and weak closure Theorem: II.

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Abstract

Given positive integers (or ∞) $r \geq 2$ and $m \geq 1$, we construct an ergodic automorphism T with rank r and $\# \frac{C(T)}{wcl\{T^n; n \in \mathbb{Z}\}} = m$. Moreover, $wcl\{T^n; n \in \mathbb{Z}\}$ is uncountable.

Contents

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1 Introduction.

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and let $C(T)$ be the metric centralizer of T. The Weak Closure Theorem [Kin1] asserts that $C(T) = wcl\{T^n, n \in \mathbb{Z}\}$, whenever $r(T) = 1$, where $r(T)$ is the rank of T. The natural question of the existence of a relationship between $r(T)$ and the cardinality $q(T)$ of the quoutient group $\frac{C(T)}{wcl\{T^n; n\in\mathbb{Z}\}}$ in the general case arises.

In [BuKwSi] a class $\mathcal{A}$ of group extensions of rank one transformations is defined such that $q(T) \geq r(T)$ for $T \in \mathcal{A}$. Moreover, the difference $q(T) - r(T)$ takes arbitrarily large positive values and an uncountable $wcl\{T^n, n \in Z\}$ can be obtained. The reverse inequality $q(T) \leq r(T)$ holds for $T \in \mathcal{M}$ (the mixing automorphisms, [Kin2]) and for $T \in \mathcal{B}$, where $\mathcal{B}$ is the class of automorphisms defined in [ChKaMFRa]. More precisely, if $T \in \mathcal{B}$ then $q(T) = 2$ and $r(T)$ can be arbitrarily large [BuKwSi]. Each automorphism $T \in \mathcal{B}$ has a discrete part in its spectrum (from this point of view the classes $\mathcal{M}$ and $\mathcal{B}$ are far from each other). However for $T \in \mathcal{B}$ or $T \in \mathcal{M}$ the powers $\{T^n, n \in Z\}$ of T form an isolated set in $C(T)$. Therefore in both cases one has $wcl\{T^n, n \in Z\} = \{T^n, n \in Z\}$. So it still remains interesting to find examples of automorphisms T such that $q(T) \leq r(T)$, $r(T) - q(T)$ is arbitrarily large and $wcl\{T^n, n \in Z\}$ is uncountable.

In this paper we obtain stronger results. We construct a class of ergodic automorphisms T such that $wcl\{T^n, n \in Z\}$ is uncountable, and $r(T) = r$, $q(T) = m$ for arbitrary $2 \leq r \leq \infty$ and $1 \leq m \leq \infty$, $(r, m) \neq (\infty, \infty)$. Our examples lie in the class of group extensions determined by r-Toeplitz sequences. The investigation of ergodicity and the metric centralizer relies on Newton's functional equation [New] and is carried out partially on a metric group extension representation of the system. Investigating the rank (and partly the centralizer too) we use a shift representation of those extensions.

2 Preliminaries.

2.1 Notations and definitions.

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue space and T a measure-preserving invertible ergodic transformation of $(X, \mathcal{B}, \mu)$. By the centralizer (metric) of T we mean the set of all measure-preserving automorphisms of $(X, \mathcal{B}, \mu)$ which commute with T and we denote it by $C(T)$. Then $C(T)$ is a topological group with the standard composition of the transformations and with a topology (called the weak topology) defined as follows: $\{S_n\}_{n\in\mathbb{N}} \in C(T)$ converges to $S \in C(T)$ if for every $A \in \mathcal{B}$

$$\mu(S_n A \triangle SA) \longrightarrow 0.$$

We shall indicate this convergence by $S_n \rightharpoonup S$.

With this topology, $C(T)$ is metric, complete. By $wcl\{T^n, n \in \mathbb{Z}\}$ we mean the weak closure of the powers of T in $C(T)$. We say that a sequence of sets $A_1, \ldots, A_k \in \mathcal{B}$ is a T-tower if these sets are pairwise disjoint and $TA_i = A_{i+1}, i = 1, \ldots, k-1$.

An ε-partition of X is a finite collection of measurable disjoint sets which covers X up to ε in measure. The rank of a dynamical system $(X, \mathcal{B}, \mu, T)$ is the smallest positive integer $r = r(T)$ such that there exists a sequence (P_n), each P_n an ε_n-partition, $\varepsilon_n \downarrow 0$, such that P_n (as a set) is a union of r T-towers. If such a positive integer does not exist then we say that $r(T) = \infty$.

Suppose now that G is a compact metric abelian group and $\varphi : X \longrightarrow G$ is a measurable function which we will call a cocycle. The G-extension of $(X, \mathcal{B}, \mu, T)$ given by the cocycle φ is the dynamical system $\mathcal{X}_\varphi = (X \times G, \mathcal{B} \times \mathcal{B}_G, \mu \times \nu, T_\varphi)$, where $\mathcal{B}_G$ is the Borel σ-algebra in G, ν is the normalized Haar measure on G and

$$T_\varphi(x,g) = (Tx, g + \varphi(x))$$

for $x \in X, g \in G$. It is well known [Par] that for ergodic $(X, \mathcal{B}, \mu, T)$

Theorem A T_φ *is ergodic iff the functional equation*

$$\frac{f(Tx)}{f(x)} = \gamma(\varphi(x)) \tag{1}$$

has no measurable solutions $f : X \longrightarrow K$ *for any nontrivial character* γ *of* G *(*K *is the unit complex circle).*

It is known (see [New] for the definition) that if $(X, \mathcal{B}, \mu, T)$ is a canonical factor of T_φ (for example if T is with discrete spectrum) then, assuming that T_φ is ergodic, $C(T_\varphi)$ is given by the triples (S, f, τ), where $S \in C(T),\ f : X \to G$ is measurable and τ is a group automorphism of G such that

$$f(Tx) - f(x) = \varphi(Sx) - \tau(\varphi(x)). \tag{2}$$

This means that every element $R \in C(T_\varphi)$ is of a form

$$R(x,g) = (Sx, \tau(g) + f(x)). \tag{3}$$

In such a case we write $R \sim (S, f, \tau)$. The following property is proved in [LeLi] and [LeLiTh], using Theorem A.

Theorem B *If* $R_n, R \in C(T_\varphi)$ *and* $R_n \sim (S_n, f_n, id), R \sim (S, f, id)$ *then* $R_n \rightharpoonup R$ *iff* $S_n \rightharpoonup S$ *and* $f_n \longrightarrow f$ *in measure* μ*.*

Let $\sigma_a : X \times G \longrightarrow X \times G$ be given by the formula

$$\sigma_a(x,g) = (x, g+a), \quad a \in G. \tag{4}$$

Then $\sigma_a \in C(T_\varphi)$, $\sigma_a \sim (id, a, id)$. For every integer $n, (T_\varphi)^n$ is given by the formula

$$(T_\varphi)^n(x,g) = (T^n x, g + \varphi^{(n)}(x)), \tag{5}$$

where

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \ldots + \varphi(T^{n-1}x), & \text{if } n \geq 0 \\ -\varphi(T^{-1}x) - \ldots - \varphi(T^n x), & \text{if } n < 0 \end{cases} \tag{6}$$

Then it follows from Theorem B that

Corollary 1 $(T_\varphi)^{n_k} \rightharpoonup \sigma_a$ *in* $C(T_\varphi)$ *iff* $T^{n_k} \rightharpoonup id$ *in* $C(T)$ *and* $\varphi^{(n_k)} \longrightarrow a$ *in measure.*

2.2 Sequences and blocks.

A finite sequence $B = (B[0], \ldots, B[k-1]), B[i] \in G, 0 \leq i \leq k-1, k \geq 1$, is called a block over G. The number k is called the length of B and is denoted by $|B|$. If $C = (C[0], \ldots, C[n-1])$ is another block then the concatenation of B and C is the block

$$BC = (B[0], \ldots, B[k-1], C[0], \ldots, C[n-1]).$$

Inductively we define the concatenation of an arbitrary number of blocks. By $B_g, g \in G$, we will denote the block

$$B_g = (B[0] + g, \dots, B[k-1] + g)$$

and by $B[i,s]$ $(0 \le i \le s \le k-1)$ the block

$$B[i,s] = (B[i], \dots, B[s]).$$

Assume that

$$B = B(0) \dots B(r-1)$$

is a concatenation of r blocks $B(0), \dots, B(r-1)$ having the same lengths and

$$C = C[0] \dots C[rm-1]$$

for some $m \ge 1$. We define the product $B \overset{r}{\times} C$ of B and C as follows:

$$B \overset{r}{\times} C = \tag{7}$$

$$B_{C[0]}(0) \dots B_{C[r-1]}(r-1) B_{C[r]}(0) \dots B_{C[2r-1]}(r-1) B_{C[r(m-1)]}(0) \dots B_{C[rm-1]}(r-1).$$

Then

$$|B \overset{r}{\times} C| = \frac{|B||C|}{r} = |B(i)|rm, \text{ for every } i = 0, \dots, r-1.$$

Let Ω by the space of all *bi*-infinite sequences over G. If $\omega \in \Omega$ or ω is a one-sided infinite sequence over G then $\omega[i,s]$, $i \le s$, denotes the block $(\omega[i], \dots, \omega[s])$. A block B is said to occur at place i in ω (resp. in a block C, $|C| = n$, if $|B| \le n$) if $\omega[i, i + |B| - 1] = B$ (resp. $C[i, i + |B| - 1] = B$). The frequencies of B in C or ω are the numbers

$$fr(B,C) = |C|^{-1} \#\{0 \le i \le |C| - |B|;\ B \text{ occurs at place } i \text{ in } C\},$$

$$fr(B,\omega) = \lim_{s \to \infty} fr(B, \omega[0, s-1]),$$

if this limit exists.

For an infinite subsequence of ω, $E = \{\omega[n], n \in I \subset \mathbb{Z}\}$ (resp. $E = \{\omega[n], n \in I \subset \mathbb{N}\}$), we call the density of E the density of the set I in $\mathbb{Z}$ (resp. in $\mathbb{N}$), whenever it exists. Let $\delta > 0$. We say that B δ-occurs at place i in C (resp. in ω) if

$$d(B, C[i, i + |B| - 1]) < \delta \ \text{(resp. } d(B, \omega[i, i + |B| - 1]) < \delta),$$

where

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = n^{-1} \#\{i;\ x_i \neq y_i\}$$

(d is called the normalized Hamming distance or d-bar distance between sequences). We will say also that B δ-occurs on the fragment $\omega[i, i + |B| - 1]$ of ω.

We will use the following elementary properties of the distance d;

$$d(B \overset{r}{\times} C, B \overset{r}{\times} D) = d(C, D) \ \text{(see (7))}, \tag{8}$$

$$d(B_g, C_g) = d(B, C), \tag{9}$$

$$(10)\qquad d(A_1A_2, B_1B_2) = \frac{|A_1|}{|A_1| + |A_2|} d(A_1, B_1) + \frac{|A_2|}{|A_1| + |A_2|} d(A_2, B_2),$$

where $|A_1| = |B_1|, |A_2| = |B_2|$.

If $D_1 \subset D$ (D_1 is a subblock of D) and $C_1 \subset C$, $|D_1| = |C_1|$, both appearing in the corresponding same positions, then

$$(11)\qquad d(D, C) \geq \frac{|D_1|}{|D|} d(D_1, C_1).$$

$$(12)\qquad d(A_1A_2 \dots A_s, B_1B_2 \dots B_s) = \frac{1}{s}\sum_{i=1}^{s} d(A_i, B_i)$$

if $|A_1| = |A_2| = \dots = |A_s| = |B_1| = \dots = |B_s|$.

By T_σ we denote the left shift homeomorphism of Ω. If $\omega \in \Omega$ then $O(\omega)$ denotes the T_σ-orbit of ω and Ω_ω the T_σ-orbit closure of ω in Ω. The T_σ-orbit closure Ω_ω is well-defined if ω is a one-sided sequence. Namely, we first let $\diamondsuit \notin G$ be an additional symbol. Then we let $\omega^\diamondsuit$ denote the bi -infinite sequence which agrees with ω at positive coordinates and has only squares appearing at the negative ones. Then we say that a *bi*-infinite y belongs to Ω_ω if there exists $n_i \to +\infty$ such that $T_\sigma^{n_i}\omega \to y$ in Ω (the convergence is for all coordinates of y, and the limiting element y does not contain any more squares). The topological flow $(\Omega_\omega, T_\sigma)$ is called minimal if there is no non -trivial closed and T_σ-invariant subset of Ω_ω. We say that $(\Omega_\omega, T_\sigma)$ is uniquely ergodic if there is a unique borelian normalized T_σ-invariant measure μ_ω on Ω_ω. Then $(\Omega_\omega, T_\sigma)$ is said to be strictly ergodic if it is minimal and uniquely ergodic. Suppose $(\Omega_\omega, T_\sigma)$ is strictly ergodic. The unique T_σ-invariant measure μ_ω is determined by the condition

$$\mu_\omega(B) = fr(B, \omega)$$

for each block B. In this case the definition of the rank can be expressed as follows.

We say that $(\Omega_\omega, T_\sigma, \mu_\omega)$ is of rank at most r if for any $\delta > 0$ and every n, there exist r blocks $B_1, \dots, B_r$, $|B_i| \geq n$, such that for all N large enough, for any $s \in \mathbb{N}$, the fragment $\omega[s, s+N-1]$ has a form

$$\omega[s, s+N-1] = \varepsilon_1 W_1 \varepsilon_2 W_2 \dots \varepsilon_k W_k \varepsilon_{k+1},$$

where $|\varepsilon_1| + \dots + |\varepsilon_k| + |\varepsilon_{k+1}| < \delta N$ and the distance d between W_j and some B_m, $j = 1, \dots, k$, $1 \leq m \leq r$, is less than δ. The system $(\Omega_\omega, T_\sigma, \mu_\omega)$ is of rank r if it is of rank at most r and not of rank at most $r-1$.

2.3 Adding machines and r-Toeplitz cocycles.

Now, let $T : (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ be a $\{p_t\}$-adic adding machine i.e.

$$p_{t+1} = \lambda_{t+1} p_t, \quad \lambda_0 = p_0, \quad \lambda_t \geq 2 \text{ for } t \geq 0,$$

$$X = \{x = \sum_{t=0}^{\infty} q_t p_{t-1};\ 0 \leq q_t \leq \lambda_t - 1,\ \ p_{-1} = 1\}$$

is the group of $\{p_t\}$-adic integers and $Tx = x + \hat{1}$, where

$$\hat{1} = 1 + 0p_1 + 0p_2 + \dots\ .$$

The space X has a standard sequence $\{\xi_t\}_{t\geq 0}$ of T-towers. Namely

$$\xi_t = (D_0^t, \ldots, D_{p_t-1}^t),$$

where

$$D_0^t = \{x \in X;\ q_0 = \ldots = q_t = 0\},\ D_s^t = T^s(D_0^t)$$

for $s = 1, \ldots, p_t - 1$. We have $X = \bigcup_{i=0}^{p_t-1} D_i^t$. Then ξ_{t+1} refines ξ_t and the sequence of partitions $\{\xi_t\}_{t\geq 0}$ converges to the point partition.

We will define a special class of cocycles $\varphi : X \longrightarrow G$ that are determined by Toeplitz sequences over G.

Let $r \geq 2$ be an integer, and assume that $b^0, b^1, \ldots$ are finite blocks over G with $|b^t| = \lambda_t r,\ \lambda_t \geq 2$, such that $b^t[0, r-1] = (\underbrace{0, \ldots, 0}_{r-times})$. We shall introduce a particular sequence (p_t), and some new blocks (B^t).

We can write

$$b^t = b^t(0) \ldots b^t(r-1),\ \ |b^t(i)| = \lambda_t,\ i = 0, \ldots, r-1. \tag{13}$$

Define another sequence of blocks $\{B^t\}$ letting

$$B^0 = b^0,\ \ B^{t+1} = B^t \overset{r}{\times} b^{t+1},\ \ t \geq 0. \tag{14}$$

Then we have

$$|B^t| = rm_t = p_t;\ \ m_t = \lambda_0 \ldots \lambda_t \tag{15}$$

and we can represent B^t as

$$B^t = B^t(0) \ldots B^t(r-1),\ \ |B^t(i)| = m_t,\ \ i = 0, \ldots, r-1. \tag{16}$$

Moreover

$$B^{t+1}[0, p_t - 1] = B^t. \tag{17}$$

Now we can define a cocycle φ by

$$\varphi(x) = B^t[i+1] - B^t[i] \tag{18}$$

if $x \in D_i^t$ except of $i = m_t - 1, 2m_t - 1, \ldots, p_t - 1$. Let us observe that φ is well defined. Such a cocycle is called r-Toeplitz cocycle. For every $t \geq 0$, φ is constant on the levels of ξ_t except of r levels.

The sequence $\{B^t\}_{t\geq 0}$ determines a one-sided sequence ω as follows:

$$\omega[0, p_t - 1] = B^t,\ \ t = 0, 1, \ldots. \tag{19}$$

The condition (17) guarantees that ω is well defined.

It is not hard to show that the condition

$$fr(g, b^t) \geq \rho > 0 \qquad \text{(if } G \text{ is finite)} \tag{20}$$

for every $g \in G$ and $t = 0, 1 \ldots,$ implies that the system $(\Omega_\omega, T_\sigma)$ is strictly ergodic. Then using (19), (20), and arguments as in [Lem], we deduce that the dynamical systems $(\Omega_\omega, T_\sigma, \mu_\omega)$ and $(X \times G, T_\varphi, \mu \times \nu)$ are metricaly isomorphic when T_φ is ergodic.

The group extensions defined by r-Toeplitz cocycles shall be called r-Toeplitz extensions.

In the sequel we will write

$$\omega = b^0 \overset{r}{\times} b^1 \overset{r}{\times} b^2 \overset{r}{\times} \dots \ .$$

Except of ω we need the sequences ω_t, $t \geq 0$, defined by

$$\omega_t = b^t \overset{r}{\times} b^{t+1} \overset{r}{\times} \dots \ . \tag{21}$$

3 Examples of r-Toeplitz extensions.

In this part, given $r \geq 2$ and $m \geq 1$, we define r-Toeplitz group extensions having cardinality of the quotient group $C(T_\varphi)/wcl\{T_\varphi^n; n \in \mathbb{Z}\}$ equal to m.

3.1 The case $r \geq 2,\ m \geq 2$.

Let $G = \mathbb{Z}/m\mathbb{Z} = \{0, \dots, m-1\}$. Define

$$F^{(i)} = \overbrace{00\dots0}^{r(2^{i+2}-1)}\ \overbrace{\underbrace{0\dots0\ 1}_{i+1}\ 0\dots0}^{r},\ i = 0, \dots, r-1;$$

$$H^{(i)} = F_0^{(i)} F_1^{(i)} \dots F_{m-1}^{(i)}.$$

We have $|H^{(i)}| = mr2^{i+2}$. Next define

$$b^t(0) = \underbrace{H^{(0)}H^{(0)}\dots H^{(0)}}_{x_0-times}$$

$$b^t(1) = \underbrace{H^{(1)}H^{(1)}\dots H^{(1)}}_{x_1-times}$$

$$\vdots$$

$$b^t(r-1) = \underbrace{H^{(r-1)}H^{(r-1)}\dots H^{(r-1)}}_{x_{r-1}-times}$$

where

$$x_i = 2^{t+r-1-i},\ \ 0 \leq i \leq r-1,$$

and

$$b^t = b^t(0)\dots b^t(r-1),\ \ t \geq 0.$$

Then we have

$$\lambda_t = |b^t(i)| = mr2^{t+r+1},\ \text{for}\ i = 0, \dots, r-1 \quad (\text{see (13)})$$

and

$$|b^t| = mr^2 2^{t+r+1}.$$

Now define the blocks B^t, $t \geq 0$, by (14) and the cocycle φ by (18). Then from (15)

$$p_t = |B^t| = m^{t+1}r^{2t}2^{r+1}(2^{t+1}-1),\ \ t \geq 0.$$

3.2 The case $r \geq 2,\ m = 1.$

Let $G = \mathbb{Z}/n\mathbb{Z} = \{0, \dots, n-1\},\ \ n \geq 4$. Then define

$$F^{(i)} = \overbrace{00\dots0}^{3r}\overbrace{0\dots\underbrace{1}_{i+1}0\dots0}^{r},$$

$$H^{(i)} = F_0^{(i)}F_1^{(i)}\dots F_{n-1}^{(i)},$$

and

$$b^t(i) = \underbrace{H^{(i)}H^{(i)}\dots H^{(i)}}_{x-times},\quad x = 2^t.$$

Next set

$$b^t = b^t(0)\dots b^t(r-1),$$

$$B^t = b^0 \overset{r}{\times} b^1 \overset{r}{\times} \dots \overset{r}{\times} b^t,\ \ t \geq 0$$

and define φ by (18). In this case we have

$$\lambda_t = rn2^{t+2} = |b^t(i)|, \quad |b^t| = r^2n2^{t+2},\ \text{for } i = 0, 1\dots, r-1 \text{ and } t \geq 0.$$

3.3 Ergodicity and the metric centralizer.

Theorem 1 *T_φ is ergodic.*

Proof. We will prove ergodicity of T_φ in both cases 3.1 and 3.2. Assume that there exists a measurable function $f : X \longrightarrow K$ satisfying (1). Then (see (5), (6))

$$\frac{f(T^n x)}{f(x)} = \gamma(\varphi^{(n)}(x)) \tag{22}$$

for μ-a.e. $x \in X$ and every $n \in \mathbb{Z}$.

In particular (22) holds for $n = p_t,\ t = 0, 1\dots$. The measurability of f and the fact that $\xi_t \longrightarrow \varepsilon$ (the partition into points) in X imply

$$\gamma(\varphi^{(p_t)}(x)) = 1 \tag{23}$$

except of a subset of measure ε_t and $\varepsilon_t \longrightarrow 0$.

Let $x \in D_j^{t+1},\ \ 0 \leq j \leq p_{t+1} - 1$. We can represent j as

$$j = up_t + vm_t + \rho, \tag{24}$$

where $0 \leq u \leq \lambda_{t+1} - 1,\ \ 0 \leq v \leq r-1,\ \ 0 \leq \rho \leq m_t - 1$ (see (15)).

It follows from (18) (with $t := t+1$) that

$$\varphi^{(p_t)}(x) = B^{t+1}[j + p_t] - B^{t+1}[j] \tag{25}$$

except j for which $u = u_1 = \frac{\lambda}{r} - 1, \dots, u = u_r = \frac{r\lambda}{r} - 1 = \lambda - 1, \lambda = \lambda_{t+1}$. At the same time we have

$$B^{t+1}[j] = b[ur + v] + B^t(v)[\rho], \quad b = b^{t+1} \quad \text{(see (14), (16))}.$$

Then (25) can be rewritten as

$$\varphi^{(p_t)}(x) = b[(u+1)r + v] - b[ur + v], \qquad u \neq u_1, \dots, u_r. \tag{26}$$

The last equality and (23) imply that

$$\gamma(c[q]) = 1, \quad (q = ur + v) \tag{27}$$

for $q \in V_t \subset \{0, 1, \ldots, r\lambda_{t+1} - 1\}$ $\frac{\# V_t}{r\lambda_{t+1}} \geq 1 - \varepsilon_t - \frac{2}{\lambda_{t+1}}$, where $c = c^t$ is given by

$$c[q] := b[q + r] - b[q], \quad q = 0, \ldots, r\lambda - r - 1.$$

Further the blocks $c = c^t$ have the following forms:

$$c = \overbrace{E^{(0)} \ldots E^{(0)}}^{(mx_0-1)-times} L^{(0)} \overbrace{E^{(1)} \ldots E^{(1)}}^{(mx_1-1)-times} L^{(1)} \ldots \overbrace{E^{(r-1)} \ldots E^{(r-1)}}^{(mx_{r-1}-1)-times} \tag{28}$$

where

$$E^{(0)} = \overbrace{0 \ldots 0}^{2r} \overbrace{10 \ldots 0}^{r} \overbrace{01 \ldots 1}^{r}, \quad |L^0| = r,$$

$$E^{(1)} = \overbrace{0 \ldots 0}^{6r} \overbrace{010 \ldots 0}^{r} \overbrace{101 \ldots 1}^{r}, \quad |L^1| = r,$$

$$\vdots$$

$$E^{(r-1)} = \overbrace{0 \ldots 0}^{(2^{r+1}-2)r} \ \overbrace{0 \ldots 01}^{r} \overbrace{1 \ldots 10}^{r}, \quad |L^{(r-2)}| = r,$$

in the case 3.1. In the case 3.2 we have

$$c = \overbrace{E^{(0)} \ldots E^{(0)}}^{(nx-1)-times} L^{(0)} \overbrace{E^{(1)} \ldots E^{(1)}}^{(nx-1)-times} L^{(1)} \ldots \overbrace{E^{(r-1)} \ldots E^{(r-1)}}^{(nx-1)-times} \tag{29}$$

where

$$E^{(0)} = \overbrace{0 \ldots 0}^{2r} \overbrace{10 \ldots 0}^{r} \overbrace{01 \ldots 1}^{r}, \quad |L^0| = r,$$

$$E^{(1)} = \overbrace{0 \ldots 0}^{2r} \overbrace{010 \ldots 0}^{r} \overbrace{101 \ldots 1}^{r}, \quad |L^1| = r,$$

$$\vdots$$

$$E^{(r-1)} = \overbrace{0 \ldots 0}^{2r} \overbrace{0 \ldots 01}^{r} \overbrace{1 \ldots 10}^{r}, \quad |L^{(r-2)}| = r.$$

In both cases 1 appears in c with frequency $> \frac{1}{r2^{r+2}}$ for each $t \geq 0$. Then (27) implies $\gamma(1) = 1$ so γ is trivial. We have proved that T_φ is ergodic. ■

3.4 The centralizer of T_φ.

The p_t-adic adding machine $(X, \mathcal{B}, \mu, T)$ is a canonical factor of the group extension $(X \times G, \mathcal{B} \times \mathcal{B}_G, \mu \times \nu, T_\varphi)$. Then $C(T_\varphi)$ is described in 2.1. We can distinguish the following subgroups of $C(T_\varphi)$:

$$C_1 = wcl\{T_\varphi^n; n \in \mathbb{Z}\}$$

$$C_2 = \{\sigma_a \circ \widetilde{S};\ \widetilde{S} \in C_1 \text{ and } a \in G\},$$

$$C_3 = \{R \sim (S, f, \tau);\ \tau = id\}.$$

Of course C_1, C_2, C_3 are closed subgroups of $C(T_\varphi)$ and

$$C_1 \subset C_2 \subset C_3 \subset C(T_\varphi).$$

We prove in Lemmas 1 and 2 that $C(T_\varphi)$ reduces to C_2 when φ is the r-Toeplitz cocycle defined in 3.1 or in 3.2.

In the sequel n means the same n as the one defined in 3.2 if this case is considered, and $n := m$ if the case 3.1 is considered.

Lemma 1 $C(T_\varphi) = C_3$.

Proof. Take R as in (3). Then the triple (S, f, τ) satisfies (2). Putting $x := Tx, \dots, T^{p_t-1}x$ in (2) and summing we obtain

$$f(T^{p_t}x) - f(x) = \varphi^{(p_t)}(Sx) - \tau(\varphi^{(p_t)}(x)) \tag{30}$$

for μ-a.e. $x \in X$ and each $t \geq 0$. Using the same arguments as in the proof of Theorem 1 we get from (30)

$$\varphi^{(p_t)}(Sx) - \tau(\varphi^{(p_t)}(x)) = 0 \tag{31}$$

for $x \in X_t$ and $\mu(X_t) \longrightarrow 1$.

Further we know [New] that there exists $q_0 \in X$ such that

$$S(x) = x + g_0, \quad x \in X.$$

Let

$$g_0 = \sum_{t=0}^{\infty} u_t p_{t-1}, \qquad 0 \leq u_t \leq \lambda_t - 1,\ t \geq 1 \text{ and } 0 \leq u_0 \leq \lambda_0 r - 1.$$

Fix t and consider (31) on the tower ξ_{t+1}. Let $j_t = \sum_{j=0}^{t} u_j p_{j-1}$. Then (see (24))

$$j_t = v_0 m_t + \rho_0, \qquad j_{t+1} = u_0 p_t + v_0 m_t + \rho_0, \quad u_0 = u_{t+1}.$$

If $x \in D_j^{t+1}$, $0 \leq j \leq p_{t+1} - 1$, then $Sx \in D_{j+j_{t+1}}^{t+1}$, where $j + j_{t+1}$ is taken mod p_{t+1}. We can write

$$j + j_{t+1} = \bar{u} p_t + \bar{v} m_t + \bar{\rho}, \quad 0 \leq \bar{u} \leq \lambda - 1,\ 0 \leq \bar{v} \leq r - 1,\ 0 \leq \bar{\rho} \leq m_t - 1.$$

Let us denote (use (24) for j)

$$q_0 = \begin{cases} u_0 r + v_0 & \text{if} \quad \rho = 0, \dots, m_t - \rho_0 - 1, \\ u_0 r + v_0 + 1 & \text{if} \quad \rho = m_t - \rho_0, \dots, m_t - 1, \end{cases}$$

and $q = ur + v$, $\bar{q} = \bar{u} r + \bar{v}$. Then $\bar{q} = q + q_0 \pmod{r\lambda_{t+1}}$. Thus (26) and (31) give

$$c[q + q_0] = \tau(c[q]) \text{ if } q \in V_t \subset \{0, 1, \dots, r\lambda_{t+1} - 1\} \tag{32}$$

and $\frac{1}{\lambda_{t+1}} \# V_t \longrightarrow 1$. Analysing the sequences (28) and (29) it is easy to observe that they do not satisfy (32) with any q_0 whenever $\tau \neq id$ (i.e. $\tau(1) \neq 1$). The Lemma is proved. ■

Lemma 2 $C(T_\varphi) = C_2$.

Proof. Let $R \sim (S, f, id) \in C_3$. Then (32) means

$$c[q + q_0] = c[q],\ q \in V_t.$$

The last condition implies

$$q_0(t) = q_0 = 2^{r+1} rmw,\ \ w = w_t, \tag{33}$$

in the case 3.1 and

$$q_0(t) = q_0 = 4rnw,\ \ w = w_t, \tag{34}$$

in the case 3.2, where $0 \le w \le r2^{t+1} - 1$ (see again (28) and (29)). Moreover

$$\min(\frac{q_0(t)}{\lambda_{t+1}}, 1 - \frac{q_0(t)}{\lambda_{t+1}}) \longrightarrow 0.$$

The above condition implies

$$\min(\frac{j_t}{p_t}, 1 - \frac{j_t}{p_t}) \longrightarrow 0.$$

Assume that $\frac{j_t}{p_t} \longrightarrow 0$ along some subsequence of t. It follows from the definition of the p_t-adic adding machine that

$$T^{j_t} \rightharpoonup S. \tag{35}$$

Now we will prove that there exists $a \in G$ such that

$$\varphi^{(j_t)} \longrightarrow f + a \tag{36}$$

in measure μ.

The function f satisfies the condition (see (2) with $\tau = id$)

$$f(Tx) - f(x) = \varphi(Sx) - \varphi(x).$$

The measurability of f and $\xi_t \longrightarrow \varepsilon$ imply that there exists $a_t \in G$ such that the functions f_t defined by

$$f_t(y) = a_t + \varphi^{(i)}(Sx) - \varphi^{(i)}(x),\ \ y \in D_i^t,\ y = T^i x,\ x \in D_0^t, \tag{37}$$

$$i = 0, \ldots, p_t - 1,$$

satisfy the condition

$$f_t \longrightarrow f \text{ in measure } \mu.$$

We can assume that $a_t = b$. We can rewrite (37) as

$$f_t(y) = b + \varphi^{(i)}(Sx) - \varphi^{(i)}(T^{j_t}x) + \varphi^{(i)}(T^{j_t}x) - \varphi^{(i)}(x).$$

Further we have (see (6))

$$\varphi^{(i)}(T^{j_t}x) - \varphi^{(i)}(x) = \varphi^{(j_t)}(T^i x) - \varphi^{(j_t)}(x). \tag{38}$$

Because of $j_t < m_t$ then $\varphi^{(j_t)}(x) = b_t$ for all $x \in D_0^t$. Assuming again $b_t = b_1$ we can write (38) as

$$\varphi^{(i)}(T^{j_t}x) - \varphi^{(i)}(x) = \varphi^{(j_t)}(y) - b_1$$

and (37) as

$$f_t(y) = b_2 + \varphi^{(j_t)}(y) + \varphi^{(i)}(Sx) - \varphi^{(i)}(T^{j_t}x). \tag{39}$$

Assume that

$$x \in D^{t+1}_{up_t}, \qquad 0 \le u \le \lambda_{t+1} - 1.$$

Then

$$T^{j_t}x \in D^{t+1}_{up_t+j_t} \quad \text{and} \quad Sx \in D^{t+1}_{(u+u_0)p_t+j_t},$$

where $u_0 = \frac{q_0}{r}$.

For $i \le p_t - j_t - 1$, $i = vm_t + \rho$ and $u \ne u_1, \ldots, u_r$ we have

$$\varphi^{(i)}(T^{j_t}x) = B^{t+1}[up_t + j_t + i] - B^{t+1}[up_t + j_t] = b^{t+1}[ur + v] - b^{t+1}[ur]$$

and

$$\begin{aligned}\varphi^{(i)}(Sx) &= B^{t+1}[(u+u_0)p_t + j_t + i] - B^{t+1}[(u+u_0)p_t + j_t] \\ &= b^{t+1}[(u+u_0)r + v] - b^{t+1}[(u+u_0)r]\end{aligned}$$

Thus

$$\varphi^{(i)}(Sx) - \varphi^{(i)}(T^{j_t}x) = (b[q+q_0] - b[q]) - (b[ur+q_0] - b[ur]), \quad q = ur + v.$$

Then (33) and (34) imply

$$\varphi^{(i)}(Sx) - \varphi^{(i)}(T^{j_t}x) = 0 \tag{40}$$

except of a set of measure $\le \frac{r}{\lambda_t} + \frac{j_t}{p_t}$.

Now (39) and (40) imply (36) with $a = -b_2$. Notice that (35) and (36) and Theorem B imply

$$T^{j_t}_{\varphi} \rightharpoonup R \circ \sigma_a.$$

This proves the Lemma. ■

To prove that $\# \frac{C(T_\varphi)}{wcl\{T^n_\varphi; n \in \mathbb{Z}\}} = m$ in case 3.1 it is sufficient to show that $\sigma_a \notin C_1$ whenever $a \in \mathbb{Z}_m,\ a \ne 0$. In the case 3.2 we will prove that $\sigma_a \in C_1$ for every $a \in \mathbb{Z}_n$ what implies

$$\# \frac{C(T_\varphi)}{wcl\{T^n_\varphi; n \in \mathbb{Z}\}} = 1.$$

To do this we need estimations of the d-distance between blocks occurring in ω and $\omega_t,\ t \ge 0$.

3.5 d-bar distance between blocks.

The sequence $\omega = b^0 \overset{r}{\times} b^1 \overset{r}{\times} \ldots$ is a concatenation of the blocks of the form

$$E_k(t) = B^t \overset{r}{\times} \bar{e}_k, \ \ E^{(s)}_k(t) = B^t \overset{r}{\times} \bar{e}^{(s)}_k, \ \ k \in \mathbb{Z}_n, \ \ s = 0, \ldots, r-1,$$

where $\bar{e}_k = (\overbrace{k, \ldots, k}^{r}), \quad \bar{e}^{(s)}_k = (\overbrace{k, \ldots, k, \underbrace{k+1}_{s^{th}\text{ place}}, k, \ldots, k}^{r}).$

The sequence $\omega_t = b^t \overset{r}{\times} b^{t+1} \overset{r}{\times} \ldots$ is a concatenation of the blocks of the form

$$e_k(t) = b^t \overset{r}{\times} \bar{e}_k, \quad e^{(s)}_k(t) = b^t \overset{r}{\times} \bar{e}^{(s)}_k.$$

The blocks $E_k = E_k(t)$, $E_k^{(s)} = E_k^{(s)}(t)$ are called t-symbols and the blocks $e_k = e_k(t)$, $e_k^{(s)} = e_k^{(s)}(t)$ are called "small" t-symbols. Each fragment $\omega[kp_t, (k+1)p_t - 1]$ of $\omega, k \in \mathbb{Z}$, is a t-symbol, and $\omega_t[k\lambda_t r, (k+1)\lambda_t r - 1]$ is a "small" t-symbol. The positions $[kp_t, (k+1)p_t - 1]$ and $[k\lambda_t r, (k+1)\lambda_t r - 1]$ will be called the natural positions in ω and ω_t respectively.

We will examine d-bar distance between the blocks mentioned above or between their special fragments. In particular, we will examine the pairs

$$b_k(i)b_k(i+1), b_k(i)b_{k+1}(i+1), b_{k+1}(i)b_k(i+1),$$

for $i = 0, \dots, r-2$ and $k \in \mathbb{Z}_n$ and

$$b_k(r-1)b_k(0), \;\; b_k(r-1)b_{k+1}(0).$$

Proposition 1 *Let*

$$\left\{ \begin{array}{ll} I = b_0^t(i)[0, \lambda_t - j - 1], & j \leq \frac{1}{2}\lambda_t, \\ II = b_k^t(i')[j, \lambda_t - 1], & k \in \mathbb{Z}_n, \; i, i' = 0, \dots, r-1, \; t \geq 0. \end{array} \right. \tag{41}$$

If

$$d(I, II) < \frac{1}{r2^{r+2}} \tag{42}$$

then $i' = i$ *and*

$$j = (n-k)r2^{i+2} + anr2^{i+2}, \; a \geq 0 \textit{ if 3.1 holds,} \tag{43}$$

$$j = (n-k)r4 + anr4, \; a \geq 0, \; \textit{if 3.2 holds.} \tag{44}$$

Proof. It is easy to observe that if $i' \neq i$ or $i' = i$ and (43) (or (44) in the case 3.2) does not hold then every subblock $F_k^{(i)}$ of I differs from the corresponding fragment in II at least in one position. Since $j \leq \frac{1}{2\lambda_t}$, this would imply the converse inequality in (42). ■

In the Propositions 2-6 the blocks $b_k^t(i) = b_k(i), \; k \in \mathbb{Z}_n, \; 0 \leq i < r$, are those defined in 3.1.

Proposition 2 *Let*

$$I = b_0(0) \dots b_0(r-1)[0, r\lambda_t - j - 1],$$

$$II = b_k(0) \dots b_k(r-1)[j, r\lambda_t - 1],$$

$$j \leq \frac{1}{2} r\lambda_t, \quad k \in \mathbb{Z}_n.$$

If

$$d(I, II) < \frac{1}{r^2 2^{r+3}} \tag{45}$$

then $j \leq \frac{1}{2}\lambda_t, \; k = 0,$ *and*

$$j \equiv 0 \; (mod \; nr2^{r+1}). \tag{46}$$

Proof. If $j > \frac{1}{2}\lambda_t$ then we can find subblocks I_1 of I and II_1 of II such that II_1 is under I_1 (see Figure 1) having the form (41) with different $j's$ and with $i' \neq i$.

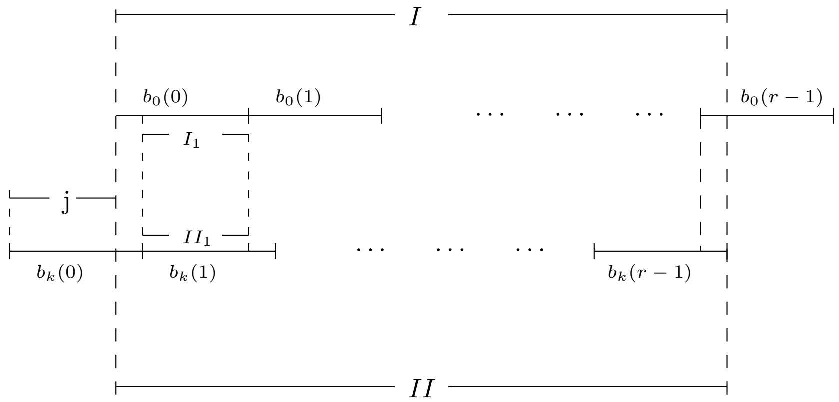

Figure 1

It follows from the Proposition 1 that $d(I_1, II_1) \geq \frac{1}{r2^{r+2}}$ and using (11) we obtain

$$d(I, II) \geq \frac{\frac{1}{2}\lambda_t}{r\lambda_t} d(I_1.II_1) \geq \frac{1}{r^2 2^{r+3}}.$$

in spite of (45). Therefore $j \leq \frac{1}{2}\lambda_t$.

It follows from (11) and (45) that

(47) $$d(I_i, II_i) < \frac{1}{r2^{r+2}} \quad \text{for } i = 0, \ldots, r-1,$$

where

$$I_i = b_0(i)[0, \lambda_t - j - 1], \ II_i = b_k(i)[j, \lambda_t - 1].$$

Then (47) implies (43) to hold for each $i = 0, \ldots, r-1$. In particular taking $i = 0, 1$ we get

$$-kr4 + 2kr4 = a_1 nr4.$$

Thus $k = 0$ in $\mathbb{Z}_n$. The Proposition is proved. ■

Proposition 3 *Let*

$$I = b_k(i) b_{k+1}(i+1)[0, 2\lambda - j - 1], \quad j \leq \frac{1}{2}\lambda; \ \lambda = \lambda_t,$$

$$II = b_{k_1}(i) b_{k_2}(i+1)[j, 2\lambda - 1], \quad i = 0, \ldots, r-2, \quad k, k_1, k_2 \in \mathbb{Z}_n$$

and $k_2 = k_1 + 1$ *or* $k_2 = k_1 - 1$.

If

(48) $$d(I, II) < \frac{1}{r2^{r+4}}$$

then

(49) $$(k_1 k_2) = (k, k+1) \ \textit{or} \ (k_1 k_2) = (k+4, k+3) \ \textit{if} \ n \geq 3$$

and

(50) $$(k_1 k_2) = (k, k+1) \ \textit{if} \ n = 2.$$

Proof. It follows from (48) and (11) that

$$d(I_1, II_1) < \frac{1}{r2^{r+2}}$$

and

$$d(I_2, II_2) < \frac{1}{r2^{r+2}},$$

where

$$I_1 = b_k(i)[0, \lambda - j - 1], \quad II_1 = b_{k_1}(i)[j, \lambda - 1],$$
$$I_2 = b_{k+1}(i+1)[0, \lambda - j - 1], \quad II_2 = b_{k_2}(i+1)[j, \lambda - 1].$$

Now, we apply the Proposition 1. It follows from (43) that

$$k - k_1 = 2(k + 1 - k_2) \pmod n.$$

The above condition implies (49) and (50). ■

Proposition 4 *Let*

$$I_k = b_k(r-1)b_k(0)[0, 2\lambda - j - 1] \text{ or } I'_k = b_k(r-1)b_{k+1}(0)[0, 2\lambda - j - 1],$$
$$II = b_{k_1}(r-1)b_{k_2}(0)[j, 2\lambda - 1], \quad k, k_1, k_2 \in \mathbb{Z}_n, \quad j \le \frac{1}{2}\lambda_t,$$

and

(51) $$k_2 = k_1 \text{ or } k_2 = k_1 + 1.$$

If

$$d(I, II) < \frac{1}{r2^{r+4}}, \quad I = I_k \text{ or } I'_k,$$

then

(52) $$k_1 = k_2 = k \text{ if } I = I_k \text{ and } k_1 = k, k_2 = k+1 \text{ if } I = I'_k$$

whenever

(53) $$(2^{r-1} - 1, n) > 1,$$

and there is a unique $l \in \mathbb{Z}_n$ such that

(54) $$\begin{cases} (k_1k_2) = (kk) \text{ or } \quad (k_1k_2) = (l, l+1) \text{ and } l \text{ satisfies} \\ l(2^{r-1} - 1) = (2^{r-1} - 1)k + 1 \text{ in } \mathbb{Z}_n \text{ if } \quad I = I_k, \\ \text{and} \\ (k_1k_2) = (k, k+1) \text{ or } \quad (k_1k_2) = (ll) \text{ and } l \text{ satisfies} \\ l(2^{r-1} - 1) = (2^{r-1} - 1)k - 1 \text{ in } \mathbb{Z}_n \text{ if } \quad I = I'_k, \end{cases}$$

whenever

(55) $$(2^{r-1} - 1, n) = 1.$$

Proof. Using the same arguments as in the proof of the Proposition 3 we obtain from (43)

$$(k_1 - k)2^{r-1} = k - k_2 \pmod n \quad \text{if } I = I_k$$

and

$$(k_1 - k)2^{r-1} = k - k_2 + 1 \pmod n \quad \text{if } I = I'_k.$$

The above, (51), (53) and (55) imply (52) and (54) respectively. ■

The next Proposition is an easy consequence of (9) and the definition of the blocks $b(0), \ldots, b(r-1)$.

Proposition 5 *Let*

$$I_l = b_l(i)[0, \lambda_t - j - 1], \quad II_k = b_k(i)[j, \lambda_t - 1],$$

$$j \le \frac{1}{2}\lambda_t, \quad k, l \in \mathbb{Z}_n, \quad 0 \le i \le r - 1.$$

If $j \equiv 0 \pmod{nr2^{r+1}}$ *and* $k \ne l$ *then*

$$d(I_l, II_k) = 1.$$

Proposition 6 *Let*

$$I = b^t \overset{r}{\times} C, \;\; II = b^t \overset{r}{\times} D[j, j + \lambda_t|D| - 1], \;\; 0 \le j \le r\lambda_t - 1,$$

where $|C| \ge 3r, |D| = |C| + r, \quad C, D \subset \omega_{t+1}$ *(see (21)) and* $C = \omega_{t+1}[pr, pr + |C| - 1], D = \omega_{t+1}[qr, qr + |D| - 1]$. *If*

$$d(I, II) < \delta, \;\; \delta < \frac{1}{3r^2 2^{r+3}}, \tag{56}$$

then either

$$j < \delta r 2^{r+1}\lambda_t \text{ and } d(C, D_1) < \delta \tag{57}$$

or

$$r\lambda_t - \delta r 2^{r+1}\lambda_t < j \le r\lambda_t \text{ and } d(C, D_1) < \delta, \tag{58}$$

where

$$D_1 = D[0, |D| - r - 1] \text{ if } j \le \frac{1}{2}r\lambda_t,$$

$$D_1 = D[r, |D| - 1] \text{ if } j > \frac{1}{2}r\lambda_t.$$

Proof. We can represent C and D as

$$C = C_1C_2 \dots C_s, \;\; D = D_1D_2 \dots D_sD_{s+1},$$

where

$$|C_1| = \ldots = |C_s| = |D_1| = \ldots = |D_s| = |D_{s+1}| = r, \quad s \ge 3,$$

and every $C_1, \ldots, C_s, D_1, \ldots, D_{s+1}$ is equal to one of the blocks $\bar{e}_k, \bar{e}_k^{(v)}, \; k \in \mathbb{Z}_n, \; v = 0, \ldots, r - 1$ (see 3.5). Assume that $j \le \frac{1}{2}r\lambda_t$. Using (12) we get

$$d(I, II) = \frac{1}{s}\sum_{p=1}^{s}(b \overset{r}{\times} C_p, A_p), \tag{59}$$

where

$$A_p = (b \overset{r}{\times} D_p)(b \overset{r}{\times} D_{p+1})[j, j + r\lambda_t - 1].$$

Then (56) implies that

$$d(b \overset{r}{\times} C_p, A_p) < \frac{1}{3r^2 2^{r+3}}$$

for at least one p. Using the same arguments as in the proof of the Proposition 2 we obtain $j \le \frac{1}{2}\lambda_t$.

Let

$$Q = \{1 \le p \le s, \;\; C_p \text{ and } D_p \text{ are equal } \bar{e}_k, \bar{e}_l \text{ for some } k, l \in \mathbb{Z}_n\}.$$

It follows from the definitions of ω, ω_t and $b^t{}'s$ that

$$\# Q \geq \frac{1}{3} s.$$

This inequality, (56), and (59), imply

$$\frac{1}{|Q|} \sum_{p \in Q} d(b \overset{r}{\times} C_p, A_p) < \frac{1}{r^2 2^{r+3}}.$$

Now we conclude that there is at least one $p \in Q$ such that

$$d(b \overset{r}{\times} C_p, A_p) < \frac{1}{r^2 2^{r+3}}.$$

It follows from the Proposition 2 that $j \equiv 0 \pmod{nr2^{r+1}}$.

Now, using (10) and (12) again we get (see Figure 2)

$$d(I, II) = \frac{1}{r} \sum_{i=0}^{r-1} \frac{1}{s}((1 - \frac{j}{\lambda_t}) \sum_{u=1}^{s} d(L_{ui}, M_{ui}) + \frac{j}{\lambda_t} \sum_{u=1}^{s} d(\bar{L}_{ui}, \bar{M}_{ui})), \tag{60}$$

where

$$L_{ui} = b^t_{C_u[i]}(i)[0, \lambda_t - j - 1], \quad M_{ui} = b^t_{D_u[i]}(i)[j, \lambda_t - 1],$$
$$\bar{L}_{ui} = b^t_{C_u[i]}(i)[\lambda_t - j, \lambda_t - 1], \quad \bar{M}_{ui} = b^t_{D_u[i]}(i+1)[0, j-1].$$

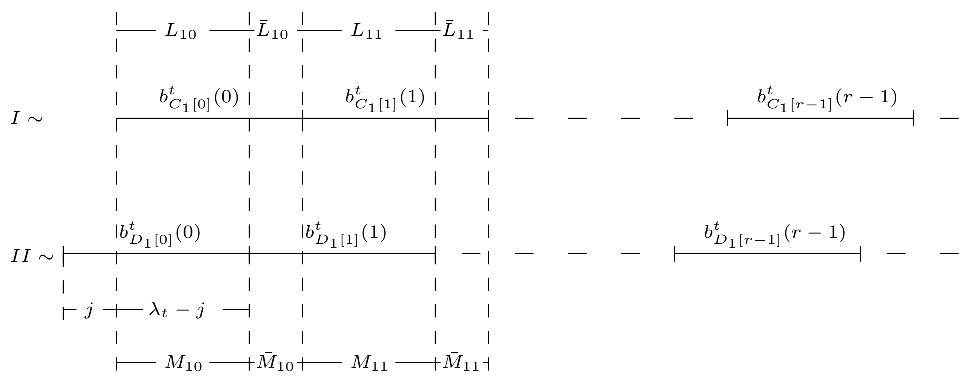

Figure 2

It is not hard to remark that if $j \neq 0$

$$d(\bar{L}_{ui}, \bar{M}_{ui}) \geq \frac{1}{r 2^{r+1}} \tag{61}$$

for every u and i, $\quad 1 \leq u \leq s, \ \ 0 \leq i \leq r-1$. Let

$$a = \#\{0 \leq k \leq |C| - 1, \ C[k] \neq D[k]\}.$$

Then using the Proposition 5, (60) and (61) we get

$$\delta > d(I, II) \geq \frac{a}{|C|}(1 - \frac{j}{\lambda_t} + \frac{j}{\lambda_t} \frac{1}{r 2^{r+1}}). \tag{62}$$

The above gives

$$\delta > \frac{a}{|C|}(1 - \frac{j}{\lambda_t}) \geq \frac{a}{|C|}\frac{1}{2}$$

and then $\frac{a}{|C|} < 2\delta$. This inequality, (56) and (62) imply

$$\delta > \frac{a}{|C|} + \frac{j}{\lambda_t}(\frac{1}{r2^{r+1}} - \frac{a}{|C|}) > \frac{a}{|C|} + \frac{j}{\lambda_t}(\frac{1}{r2^{r+1}} - 2\delta) > \frac{a}{|C|} = d(C, D_1).$$

We have obtained the second inequality of (57). To get the first inequality of (57) we use (62) to obtain

$$\delta > \frac{j}{\lambda_t}\,\frac{1}{r2^{r+1}}.$$

This implies (57). We have proved the Proposition if $j \leq \frac{1}{2}r\lambda_t$. The case $\frac{1}{2}r\lambda_t < j < r\lambda_t$ leads to (58) in a similar way. The Proposition is proved. ■

Proposition 7 *Let*

$$I = B^t \overset{r}{\times} C, \quad II = B^t \overset{r}{\times} D[j, j + m_t|D| - 1], \quad 0 \leq j \leq p_t - 1,$$

where C *and* D *satisfy the same conditions as in the Proposition 6. If*

$$d(I, II) < \delta, \quad \delta < \frac{1}{3r^2 2^{r+3}},$$

then either

$$j < \delta r 2^{r+} p_t \textit{ and } d(C, D_1) < \delta$$

or

$$p_t - \delta r 2^{r+1} p_t < j < p_t \textit{ and } d(C, D_1) < \delta$$

where

$$D_1 = D[0, |D| - r - 1] \textit{ if } j \leq \frac{1}{2}p_t,$$

and

$$D_1 = D[r, |D| - 1] \textit{ if } r > \frac{1}{2}p_t.$$

Proof. We use an induction argument and can repeat the proof of Lemma 3 from [FiKw] using (8), (9) and the Proposition 6 instead of Lemma 2 from [FeKw]. ■

3.6 d-bar distance between blocks - the case 3.2.

Using the same methods as in 3.5 we can estimate the distance between blocks $b_k^t(i)$ and $B_k^t(i),\ i = 0, \ldots, r-1,\ k \in \mathbb{Z}_n,\ t \geq 0$, defined in the case 3.2.

As an easy consequence of the Proposition 1 we get

Proposition 8 *Let*

$$I_l = b_{l_0}(0) \ldots b_{l_{r-1}}(r-1)[0, r\lambda_t - j - 1],$$

$$II_k = b_{k_0}(0) \ldots b_{k_{r-1}}(r-1)[j, r\lambda_t - 1],$$

$j \leq \frac{1}{2} r\lambda_t$, *where* $(l_0, \ldots, l_{r-1})$ *(resp.* $(k_0, \ldots, k_{r-1})$*) is of the form* $\bar{e}_l$ *or* $\bar{e}_l^{(v)}$ *(resp.* $\bar{e}_k$ *or* $\bar{e}_k^{(v')}$*),* $k, l \in \mathbb{Z}_n$ *and* $v, v' = 0, \ldots, r-1$. *If*

$$d(I_l, II_k) < \frac{1}{r^2 2^{r+3}}$$

then $j \leq \frac{1}{2}\lambda_t$ *and there is a unique* $s \in \mathbb{Z}_n$, $s = s(t)$, *such that* $l_i = k_i + s$ *for every* $i = 0, \ldots, r-1$. *Moreover* j *has a form*

$$j = (n-s)r4 + anr4, \quad a \geq 0.$$

As an analogue of the Proposition 5 we obtain

Proposition 9 *Let* I_l, II_k *be as in the Proposition 5,*

$$j \leq \frac{1}{2}\lambda_t \text{ and } j \equiv (n-s) \pmod{4rn}$$

for some $s \in \mathbb{Z}_n$. *Then*

$$d(I_l, II_k) = 1 \text{ whenever } k - l \neq s.$$

Then using the Propositions 8 and 9 we have

Proposition 10 *Let* I *and* II *be as in the Proposition 6 and*

$$|C| \geq r, \ |D| = |C| + r, \ \ C, D \subset \omega_{t+1},$$

$$C = \omega_{t+1}[pr, pr + |C| - 1], \ D = \omega_{t+1}[qr, qr + |D| - 1].$$

If

$$d(I, II) < \delta, \quad \delta < \frac{1}{3r^2 2^{r+3}},$$

then there is an unique $s \in \mathbb{Z}_n$, $s = s(t)$, *such that*

$$j < \delta r 2^{r+1}\lambda_t \text{ and } d(C, D_1) < \delta$$

or

$$r\lambda_t - \delta r 2^{r+1}\lambda_t < j \leq r\lambda_t \text{ and } d(C, D_1) < \delta$$

where $D_1 = D[0, |D| - r - 1] = C + s$ *if* $j \leq \frac{1}{2}\lambda_t r$, *and* $D_1 = D[r, |D| - 1] = C + s$ *if* $j > \frac{1}{2} r\lambda_t$.

Using arguments as in Lemma 3 in [FiKw] and the Proposition 10 we get

Proposition 11 *Let* I *and* II *be as in the Proposition 7 and* C, D *satisfy the same conditions as in the Proposition 10.*

If

$$d(I, II) < \delta, \quad \delta < \frac{1}{3r^2 2^{r+3}},$$

there exists an unique $s \in \mathbb{Z}_n$, $s = s(t)$, *such that either*

$$j < \delta r 2^{r+1} p_t \text{ and } d(C, D_1) < \delta$$

or

$$p_t \delta r 2^{r+1} p_t < j < p_t \text{ and } d(C, D_1) < \delta$$

where

$$D_1 = D[0, |D| - r - 1] + s \text{ if } j \leq \frac{1}{2} p_t$$

and

$$D_1 = D[r, |D| - 1] + s \text{ if } j > \frac{1}{2} p_t.$$

3.7 The centralizer of T_φ (continuation).

In 3.4 we have proved that $C(T_\varphi)$ consists of the elements $R\circ\sigma_a$, where R is a limit of powers of T_φ and σ_a is defined by (4), $a\in \mathbb{Z}_n$. Now we are in a position to show that $\#\frac{C(T_\varphi)}{wcl\{T_\varphi^n; n\in\mathbb{Z}\}} = \begin{cases} n & \text{in the case 3.1,} \\ 1 & \text{in the case 3.2} \end{cases}$.

Lemma 3 *If the case 3.1 holds and $\sigma_a \in C_1$ then $a=0$.*

Proof. Let us suppose that $T_\varphi^{n_s} \rightharpoonup \sigma_a,\ \ a\in\mathbb{Z}_n$. Then Corollary 1 says that $\varphi^{(n_s)} \longrightarrow a$ in measure. Let

$$\varepsilon_s = \mu\{x\in X; \varphi^{(n_s)}(x)\neq a\}. \tag{63}$$

We have $\varepsilon_s \longrightarrow 0$. Now for every s find t_s such that

$$\frac{n_s}{p_{t_s}} < \frac{\varepsilon_s}{r}. \tag{64}$$

To shorten notation we let $t:=t_s+1,\ \bar{t}:=t_s$. Take $x\in D_j^t$. Then using (18) we get

$$\varphi^{(n_s)}(x) = B^t[j+n_s]-B^t[j] \tag{65}$$

except of $j's$ satisfying $m_t-1-n_s\le j\le m_t-1,\ \ 2m_t-1-n_s\le j\le 2m_t-1,\dots,p_t-1-n_s\le j\le p_t-1$. Then (63) and (64) imply

$$\frac{1}{p_t}\#\{0\le j\le p_t-1,\ B^t[j+n_s]-B^t[j]\neq a\} < \varepsilon_s+\varepsilon_s = 2\varepsilon_s.$$

This means that

$$d(B^t[0,p_t-n_s-1], B^t_{-a}[n_s,p_t-1]) < 2\varepsilon_s.$$

We can write

$$B^t = B^{\bar{t}} \overset{r}{\times} b^t,\quad B^t_{-a} = B^{\bar{t}} \overset{r}{\times} b^t_{-a}.$$

If $\varepsilon_s < \frac{1}{6r^2 2^{r+3}}$ then we apply Proposition 7 to the blocks $I = B^{\bar{t}} \overset{r}{\times} b^t$ and $II = B^{\bar{t}} \overset{r}{\times} b^t_{-a}$. As a consequence we obtain

$$d(b^t, b^t_{-a}) < 2\varepsilon_s.$$

This equality implies (Proposition 2) $a=0$. The Lemma is proved. ■

¿From Lemmas 2 and 3 we obtain

Theorem 2 $\#\frac{C(T_\varphi)}{wcl\{T_\varphi^n, n\in\mathbb{Z}\}} = n$ *if the case 3.1 holds.*

Now, we examine the case 3.2. It follows from the definition of the blocks $b_0(i) = b_0^t(i),\ i=0,\dots,r-1,\ \ a\in\mathbb{Z}_n$ that

$$b(i)[(n-a)4r, \lambda-1] = b_a(i)[0,\lambda-(n-a)4r-1], \tag{66}$$

for every $i=0,\dots,r-1$.

Set $n_t = (n-a)4rp_{t-1}$. Then (66) implies

$$B^t(i)[j+n_t]-B^t(i)[j] = a$$

for $j=0,\dots,p_t-n_t-1$, and $i=0,\dots,r-1$. (65) and the above imply $\varphi^{(n_t)}(x)=a$ except of a set of measure $< r\frac{n_t}{p_t} \le \frac{4r^2n}{\lambda_t}$.

Hence $\varphi^{(n_t)} \longrightarrow a$ in measure which implies that $T_\varphi^{n_t} \rightharpoonup \sigma_a,\ \ a\in\mathbb{Z}_n$. We have shown that $\sigma_a\in C_1$ for every $a\in\mathbb{Z}_n$ and as a consequence of Lemma 2 we get

Theorem 3 $\# \frac{C(T_\varphi)}{wcl\{T_\varphi^n; n\in \mathbb{Z}\}} = 1$ *if the case 3.2 holds.*

Theorem 3′ $wcl\{T_\varphi^n, n \in Z\}$ is uncountable.

Proof. Let $g_0 = \sum_0^\infty u_t p_{t-1}$, $u_t = w_t(rm2^{r+1})$ in the case (3.1) and $u_t = w_t(4rn)$ in the case (3.2) $0 \le u_t \le r\lambda_t - 1$ and assume that

$$\sum_{t=0}^{\infty} \min(\frac{w_t}{r2^t}, 1 - \frac{w_t}{r2^t}) < \infty.$$

Repeating the same arguments as in Lemma 4 of [GoKwLeLi] we can construct a measurable function $f : X \longrightarrow G$ such that

$$f(Tx) - f(x) = \varphi(Sx) - \varphi(x), \text{ for a. e. } x \in X.$$

Thus the triple $R = (S, f, id) \in C(T_\varphi)$. Of course, there is a continuum of g_0's in X satisfying the above conditions. Hence $C(T_\varphi)$ is uncountable. Then Theorem 2 and 3 imply $wcl\{T_\varphi^n, n \in Z\}$ is uncountable. ■

4 Rank of T_φ is r.

In this section we use the shift representation $(\Omega_\omega, T_\sigma)$ of $(X \times \mathbb{Z}_n, T\varphi)$ (see 2.3) and the definition of rank given in 2.2.

4.1 The frequencies of t-symbols and an estimation of the rank.

Let $Fr(E, \omega)$ be the average frequency of a t-symbol E (see 3.5) appearing in ω at natural positions. Similarly, let $Fr(e, \omega_t)$ denote the average frequency of a "small" t-symbol e appearing in ω_t at natural positions. It is easy to get the following equalities;

$$(67) \qquad \begin{cases} Fr(E_k, \omega) = Fr(e_k, \omega_t) = \frac{1}{rn} \sum_{i=0}^{r-1} (1 - \frac{1}{2^{i+2}}) = \frac{1}{n}[1 - \frac{1}{r} \sum_{i=0}^{r-1} \frac{1}{2^{i+2}}] \\ \text{and} \\ Fr(E_k^{(s)}, \omega) = Fr(e_k^{(s)}, \omega_t) = \frac{1}{rn2^{s+2}}, \quad s = 0, \ldots, r-1, \ k \in \mathbb{Z}_n, \end{cases}$$

if the case 3.1 holds. In the case 3.2 we have

$$(68) \qquad \begin{cases} Fr(E_k, \omega) = Fr(e_k, \omega_t) = \frac{3}{4n}, \\ Fr(E_k^{(s)}, \omega) = Fr(e_k^{(s)}, \omega_t) = \frac{1}{4nr}, \quad k \in \mathbb{Z}_n, \ s = 0, \ldots, r-1. \end{cases}$$

Proposition 12 $r(T_\varphi) \le r$.

Proof. Consider the blocks

$$L_k^{(s)} = L_k^{(s)}(t) = B^t \overset{r}{\times} b_k^{t+1}(s), \quad s = 0, \ldots, r-1, \ t \ge 0, \ k \in \mathbb{Z}_n.$$

We have

$$E_k = L_k^{(0)} \ldots L_k^{(r-1)}, \quad E_k^{(s)} = L_k^{(0)} \ldots L_k^{(s-1)} L_{k+1}^{(s)} L_k^{(s+1)} \ldots L_k^{(r-1)}$$

for every $k \in \mathbb{Z}_n$ and $s = 0, \ldots, r-1$.

Because the blocks $E_k, E_k^{(s)}$ cover completely the sequence ω then the blocks $L_k^{(0)} \dots L_k^{(r-1)}$, $k \in \mathbb{Z}_n$, also cover ω.

We know that

$$b^{t+1}(s)[0, \lambda_{t+1} - knr2^{r+1}] = b_{-k}^{t+1}(0)[knr2^{r+1}, \lambda_{t+1} - 1],$$
$$k \in \mathbb{Z}_n, s = 0, \dots, r-1, \quad \text{if 3.1 holds},$$

and

$$b^{t+1}(s)[0, \lambda_{t+1} - knr4] = b_{-k}^{t+1}(0)[knr4, \lambda_{t+1} - 1],$$
$$k \in \mathbb{Z}_n, s = 0, \dots, r-1, \quad \text{if 3.2 holds}.$$

The last equalities imply that the block $L_0^{(s)}$ cover each block $L_k^{(s)}$, $k \in \mathbb{Z}_n$, except of a part with the length $\leq n^2 2^{r+1} p_t$ in the case 3.1 and $\leq n^2 4 p_t$ in the case 3.2, for $s = 0, \dots, r-1$. Thus the blocks $L_0^{(0)}, \dots, L_0^{(r-1)}$ cover the sequence ω except of a part with the density $\leq \frac{n^2 2^{r+1}}{\lambda_{t+1}}$ if 3.1 holds and $\leq \frac{n^2 4}{\lambda_{t+1}}$ if 3.2 holds. Simultaneously $|L_0^{(s)}(t)| \overset{t\to\infty}{\longrightarrow} \infty$. According to the definition of the rank (see 2.2) we have $r(T_\varphi) \leq r$. ■

4.2 Special subblocks of ω_t.

Fix $t \geq 0$. We distinguish special subblocks C of ω_t of the form $b^t \overset{r}{\times} \bar{C}$, where $\bar{C}$ is a strict subblock of one of the following blocks (cf. 3.5)

$$(69) \qquad \begin{cases} e_k e_k,\ e_k e_k^{(s)},\ e_k^{(s)} e_{k+1}, \quad k \in \mathbb{Z}_n,\ s = 0, \dots, r-1, \\ \text{where } e_k = e_k(t+1),\ e_k^{(s)} = e_k^{(s)}(t+1), \\ \text{if the case 3.2 is considered,} \end{cases}$$

or

$$(70) \qquad \begin{cases} e_k e_k e_k e_k,\ e_k e_k e_k e_k^{(s)},\ e_k e_k e_k^{(s)} e_{k+1},\ e_k e_k^{(s)} e_{k+1} e_{k+1},\ e_k^{(s)} e_{k+1} e_{k+1} e_{k+1}, \\ k \in \mathbb{Z}_n,\ s = 0, \dots, r-1, \\ \text{if the case 3.1 is considered.} \end{cases}$$

Notice that blocks (69) are all pairs of "small" $(t+1)$-symbols appearing in ω_{t+1}, as well as the blocks (70) are all possible quadruples of "small" $(t+1)$-symbols appearing in ω_{t+1}. Let us list the different cases we shall deal with afterwards;

A)$\bar{C} \subset b_{k_0}^{t+1}(i_0)$ for some $k_0 \in \mathbb{Z}_n$ and $i_0 = 0, \dots, r-1$ (cases 3.1 or 3.2);

B) (the case 3.2) $\bar{C} = b_{k_{i_0}}(i_0) \dots b_{k_{r-1}}(r-1) \mid b_{l_0} \dots b_{l_{i_1}}(i_1)$ where $b(i) = b^{t+1}(i)$, $i_0 > 0$, $i_1 < r-1$.
$E := (k_{i_0} \dots k_{r-1} l_0 \dots l_{i_1})$ is contained in one of the following blocks;

$$(71) \qquad \bar{e}_k \bar{e}_k, \bar{e}_k \bar{e}_k^{(s)}, \bar{e}_k^{(s)} \bar{e}_{k+1},\ \ k \in \mathbb{Z}_n,\ s = 0, \dots, r-1,$$

and $2 \leq |E| < 2r$;

B') (the case 3.1) $\bar{C} =$
$b_{k_{i_0}}(i_0)..b_{k_{r-1}}(r-1) \mid b_{u_0}(0)..b_{u_{r-1}}(r-1) \mid b_{v_0}(0)..b_{v_{r-1}}(r-1) \mid b_{l_0}(0)..b_{l_{i_1}}(i_1)$
and $E = (k_{i_0} \dots k_{r-1} \mid u_0 \dots u_{r-1} \mid v_0 \dots v_{r-1} \mid l_0 \dots l_{i_1})$, $2 \leq |E| < 4r$, $i_0 > 0$, $i_1 < r-1$, is contained in one of the blocks

$$(72) \qquad \bar{e}_k \bar{e}_k \bar{e}_k \bar{e}_k, \bar{e}_k \bar{e}_k \bar{e}_k \bar{e}_k^{(s)}, \bar{e}_k \bar{e}_k \bar{e}_k^{(s)} \bar{e}_{k+1}, \bar{e}_k \bar{e}_k^{(s)} \bar{e}_{k+1} \bar{e}_{k+1}, \bar{e}_k^{(s)} \bar{e}_{k+1} \bar{e}_{k+1} \bar{e}_{k+1}.$$

In general we can write

$$\bar{C} = \bar{C}_1\bar{C}_2\bar{C}_3 \tag{73}$$

where $\bar{C}_2$ is as in A) or as in B) (the case 3.2) or B') (the case 3.1), and

$$\begin{cases} \bar{C}_1 = b^{t+1}_{k'}(i_0-1)[l_1 r, \lambda-1], \quad \bar{C}_3 = b^{t+1}_{k''}(i_1+1)[0, l_2 r - 1], \\ 0 < l_1 \le \lambda - 1, \ 0 < l_2 \le \lambda - 1, \ \lambda = \lambda_{t+1}, \end{cases} \tag{74}$$

and $k'Ek''$ is contained in one of the blocks (71) or (72) respectively (E is defined by $\bar{C}_2$).

Then we can distinguish the next special kinds of blocks (73) for given $\delta > 0$:

G1) $\frac{|\bar{C}_1|}{|\bar{C}|} > \delta$ and $\frac{|\bar{C}_3|}{|\bar{C}|} > \delta$,

G2) $\frac{|\bar{C}_1|}{|\bar{C}|} > \delta$ and $\frac{|\bar{C}_3|}{|\bar{C}|} \le \delta$,

G3) $\frac{|\bar{C}_1|}{|\bar{C}|} \le \delta$ and $\frac{|\bar{C}_3|}{|\bar{C}|} > \delta$,

G4) $\frac{|\bar{C}_1|}{|\bar{C}|} \le \delta$ and $\frac{|\bar{C}_3|}{|\bar{C}|} \le \delta$.

4.3 $r(T_\varphi) = r$: the case 3.2.

Take $0 < \delta^2 < \frac{1}{r^2 2^{2r+3}}$.

Proposition 13 *Assume that $\bar{C}$ is as in B) and let $d(C, D) < \delta^2, D \subset \omega_t$. Then D has a form*

$$D = (b^t \overset{r}{\times} \bar{D})[j, j + |D| - 1], \ \textit{where } \bar{D} \subset \omega_{t+1} \tag{75}$$

and

$$\begin{cases} \bar{D} = b^{t+1}_{k'_{i_0}}(i_0) \dots b^{t+1}_{k'_{r-1}}(r-1) \mid b^{t+1}_{l'_0}(0) \dots b^{t+1}_{l'_{i_1}}(i_1) b^{t+1}_{l'_{i_1+1}}(i_1+1), \\ \textit{and } j < \delta^2 r 2^{r+1}\lambda_{t+1}, \ l'_{i_1+1} \in \mathbb{Z}_n \end{cases} \tag{76}$$

or

$$\begin{cases} \bar{D} \textit{ is as in (76) and} \\ j > r\lambda_{t+1} - \delta^2 r 2^{r+1}\lambda_{t+1}. \end{cases} \tag{77}$$

Moreover, there is a unique $s_0 \in \mathbb{Z}_n$ such that

$$(k'_0 \dots k'_{r-1} \mid l'_0 \dots l'_{i_1}) = (k_0 \dots k_{r-1} \mid l_0 \dots l_{i_1}) + s_0$$

if (76) holds and

$$(k'_1 \dots k'_{r-1} \mid l'_0 \dots l'_{i_1+1}) = (k_0 \dots k_{r-1} \mid l_0 \dots l_{i_1}) + s_0$$

if (77) holds.

Proof. The Proposition is an easy consequence of the Proposition 10 where t is taken instead of $t+1$ ($\delta^2 < \frac{1}{r^2 2^{2r+3}} < \frac{1}{3r^2 2^{r+3}}$). ■

Given a block $A \subset \omega$ or ω_t, $A = \omega[l, l + |A| - 1]$ we define $A(\delta)$ as $A(\delta) = \omega[l - \delta|A|, l + |A| + \delta|A| - 1]$, $\delta > 0$. The next Proposition says that if C is as in G1), G2), G3), or G4), there is a block $C' = b^t \overset{r}{\times} \widetilde{C}$ such that $\widetilde{C}$ is as in B) and either $\widetilde{C}$ contains $\bar{C}$ or $\bar{C}$ is contained in $\widetilde{C}(\delta_1)$, where $\delta_1 < \delta^2 r 2^{r+1}$.

Proposition 14 *Let* $C = b^t \overset{r}{\times} \bar{C}$ *and let* $\bar{C}$ *be as in G1), G2), G3) or G4). Assume that*

$$d(C, \omega_t[l, l + |C| - 1]) < \frac{\delta^2}{3}. \tag{78}$$

Then

$$d(C', \omega_t[l', l' + |C'| - 1]) < \delta^2$$

where $C' = b^t \overset{r}{\times} \widetilde{C}$, $\widetilde{C} \subset \omega_{t+1}$ *and*
g1) $\widetilde{C} = b_{k'}^{t+1}(i_0 - 1)\bar{C}_2 b_{k''}^{t+1}(i_1 + 1)$, $l' = l - l_1 r$ *(cf. (73), (74)), if G1) holds,*
g2) $\widetilde{C} = b_{k'}^{t+1}(i_0 - 1)\bar{C}_2$, $l' = l - l_1 r$, *if G2) holds,*
g3) $\widetilde{C} = \bar{C}_2 b_{k''}^{t+1}(i_1 + 1)$, $l' = l$, *if G3) holds,*
g4) $\widetilde{C} = \bar{C}_2$, $l' = l$, *if G4) holds.*

Proof. Consider the case G2). Then (11) and (78) imply ($C_2 = b^t \overset{r}{\times} \bar{C}_2$)

$$d(b^t \overset{r}{\times} \bar{C}_2, \omega_t[\bar{l_2}, \bar{l_2} + |C_2| - 1]) < \delta^2$$

where $\bar{l_2} = l + |b^t \overset{r}{\times} \bar{C}_1|$.

It follows from the Proposition 13 that $\omega_t[\bar{l_2}, \bar{l_2} + |C_2| - 1]$ is of the form (75). Assume that the case (76) holds. Set

$$\widetilde{C_1} = \bar{C}_1[0, |\bar{C}_1| - j - 1],$$

$$\widetilde{D_1} = \omega_{t+1}[\frac{1}{\lambda_t}(l - j), \frac{1}{\lambda_t}(l - j) + |\widetilde{C_1}| - 1] \qquad \text{(see Figure 3).}$$

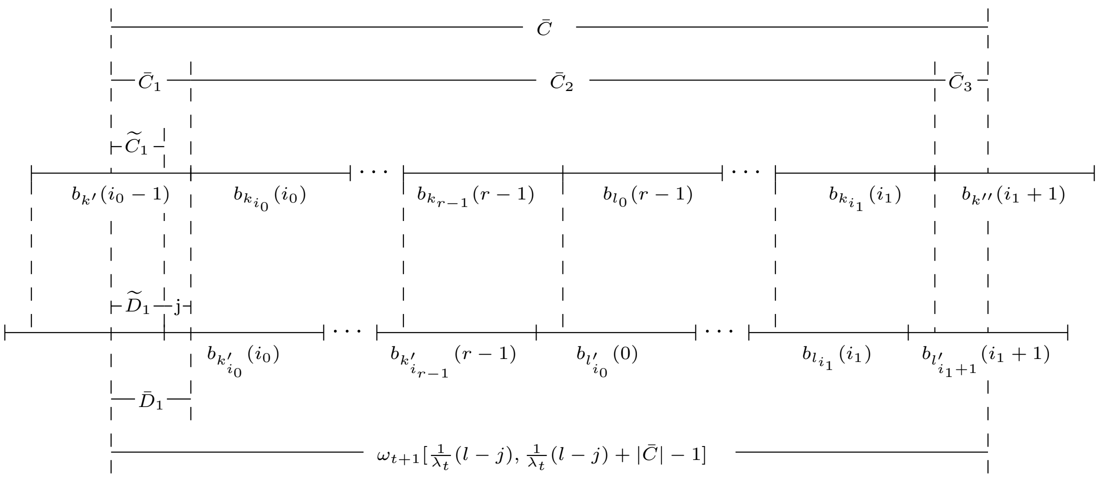

Figure 3

If follows from the Proposition 8 that

$$j \equiv (n - s_0)r4 \pmod{4nr}. \tag{79}$$

The fragment of ω_{t+1} from the left side of $b^{t+1}_{k'_0}(i_0)$ having the length λ_{t+1} is of a form $b^{t+1}_u(i_0-1)$ and either $u = k'+s_0$ or $u = k'+s_0+1$. Assume that $u = k'+s_0+1$. Then the Proposition 9 implies

$$d(\widetilde{C_1}, \widetilde{D_1}) = 1. \tag{80}$$

Let $\bar{D}_1$ denote the block $\omega_{t+1}[\frac{1}{\lambda_t}(l-j), \frac{1}{\lambda_t}(l-j)+|\bar{C}_1|-1]$ (see Figure 3). Obviously we have

$$\frac{|\bar{C}_1|}{|C|} d(\bar{C}_1, \bar{D}_1) \overset{(11),(8)}{\leq} d(C, \omega_t[l, l+|C|-1]) < \delta^2.$$

Further

$$\delta^2 > \tfrac{|\bar{C}_1|}{|C|} d(\bar{C}_1, \bar{D}_1) > \delta d(\bar{C}_1, \bar{D}_1) \overset{(11)}{\geq} \tfrac{|\widetilde{C}_1|}{|\bar{C}_1|} \delta d(\widetilde{C_1}, \widetilde{D_1}) \overset{(80)}{=} \tfrac{|\bar{C}_1|-j}{|\bar{C}_1|}\delta$$
$$= \delta(1 - \tfrac{j}{|\bar{C}_1|}) \overset{(G2)}{\geq} \delta(1 - \tfrac{j}{\delta|\bar{C}|}) \overset{(76)}{\geq} \delta(1 - \tfrac{\delta^2 r 2^{r+1}\lambda_{t+1}}{\delta|\bar{C}|}) \geq \delta(1 - \delta r 2^{r+1}),$$

because $|\bar{C}| \geq \lambda_{t+1}$.

Thus

$$1 - \delta r 2^{r+1} < \delta$$

which is in contradiction with the inequality $\delta^2 < \frac{1}{r^2 2^{2r+3}}$. We have shown $u - k' = s_0 = k'_0 - k_0$.

Now, using (79) and the definition of $b_{k'}(i_0-1)$ and $b_u(i_0-1)$ we obtain $C[v] = \omega_t[l'+v]$ for each $v = 0, \ldots, |\bar{C}_1|-1$, $l' = l - l_1 r$ (see (74)). This last equality implies g2). The proofs of the remaining cases are similar. ■

Proposition 15 *Assume that $\mathcal{F} = \{C_1, \ldots, C_d\}$, $d \leq r-1$, is a family of subblocks of ω_t such that*

$$C_j = b^t \overset{r}{\times} \bar{C}_j \text{ and each } \bar{C}_j \text{ is as in B).} \tag{81}$$

Let $\omega_t(\mathcal{F})$ be the maximal subsequence of ω_t that can be δ^2-covered by the family $\mathcal{F}$ in a disjoint way, $\delta^2 < \frac{1}{r^2 2^{2r+3}}$, and let $\bar{\omega}_t(\mathcal{F})$ be the complementary part of ω_t. Then it is an union of at least $(r-d)$ blocks $b^t \overset{r}{\times} b^{t+1}(i_j)$, $j = 1, \ldots, r-d$.

Proof. Denote by $\mathcal{F}_i$ the set of all blocks $C \in \mathcal{F}$ such that $\bar{C}$ δ^2-covers a subblock of ω_{t+1} containing one of the form

$$b^{t+1}_1(i) b^{t+1}(i+1), \; i = 0, \ldots, r-2,$$

and by $\mathcal{F}_{r-1}$ those C for which $\bar{C}$ δ^2-covers a block containing $b^{t+1}(r-1)b^{t+1}(0)$. We show that $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$ whenever $i \neq j$. Take $C \in \mathcal{F}_i, D \in \mathcal{F}_j$ and let $\bar{C}, \bar{D}$ be the blocks defined by (81), $\bar{C}$ as in B) and

$$\bar{D} = b^{t+1}_{k'_{i'_0}}(i'_0) \ldots b^{t+1}_{k'_{r-1}}(r-1) \mid b^{t+1}_{l'_0}(0) \ldots b^{t+1}_{l'_{i'_1}}(i'_1).$$

If $(i_0 \ldots (r-1) \mid 0 \ldots i_1) \neq (i'_0 \ldots (r-1) \mid 0 \ldots i'_1)$ then $C \neq D$.
If $(i_0 \ldots (r-1) \mid 0 \ldots i_1) = (i'_0 \ldots (r-1) \mid 0 \ldots i'_1)$ then using the Proposition 13 we obtain

$$(k_{i_0} \ldots k_{r-1} \mid l_0 \ldots l_{i_1}) = (k'_{i_0} \ldots k'_{r-1} \mid l'_0 \ldots l'_{i_1}) + s_0$$

for some $s_0 \in \mathbb{Z}_n$. The last condition is impossible since $i \neq j$. The Proposition follows because $\#\{\mathcal{F}_i; 0 \leq i < r\} = r$. ■

Theorem 4 $r(T_\varphi) = r$.

Proof. According to the Proposition 12 it remains to show that $r(T_\varphi) > r - 1$. Let $\frac{\delta^2}{9} < \frac{1}{r^2 2^{2r+3}}$ and let $A_1, \ldots, A_x$ be blocks occurring in ω, $|A_i| \geq p_{t_0}$ and t_0 satisfies $\frac{r}{\lambda_t} < \delta^2 r 2^{r+1}$, if $t \geq t_0,\ x \leq r-1$. For each $u = 1, \ldots, x$ there exists an unique $t = t(u)$ such that A_u contains at least one t-symbol and does not contain any $(t+1)$-symbol. Then A_u has a form

$$A_u = \widetilde{E_1}(B^{t-1} \overset{r}{\times} C_u)\widetilde{E_2}, \tag{82}$$

where $C_u \subset \omega_t$ is as in 4.2, $|C_u| = qr,\ q = q(u) \geq 1,\ E_1$ is a right-side part of a t-symbol and E_2 is a left-side part of a t-symbol. We divide the set $\{t(1), \ldots, t(x)\}$ by arithmetic order. More precisely, we put

$$\tau_1 = \max\{t(1), \ldots, t(x)\},\ T_1 = \{u; t(u) = \tau_1\},\ d_1 = \#T_1.$$

Next we define

$$\tau_2 = \max\{t(u); u \notin T_1\},\ T_2 = \{u; t(u) = \tau_2\},\ d_2 = \#T_2.$$

Similarly we define sets $T_3, \ldots, T_v$, numbers $\tau_3, \ldots, \tau_v$ and $d_3, \ldots, d_v$. We have

$$\tau_1 > \ldots > \tau_v,\ \ d_1 + \ldots + d_v = x.$$

Let

$$\mathcal{A}_p = \{A_u; u \in T_p\},\ p = 1, \ldots, v.$$

The families $\mathcal{A}_1, \ldots, \mathcal{A}_v$ are pairwise disjoint and $\cup_{p=1}^{v} \mathcal{A}_p = \{A_1, \ldots, A_x\}$.

Consider the family $\mathcal{A}_1$. Assume that

$$\mathcal{A}_1 = \{A_1, \ldots, A_{d_1}\}.$$

Then

$$C_u = b^t \overset{r}{\times} \bar{C_u}$$

and

$$\bar{C_u} \subset \omega_{t+1},\ u \in T_1,\ t = \tau_1.$$

If $d(A_u, \omega[\widetilde{l}, \widetilde{l} + |A_u| - 1]) < \frac{\delta^2}{9}$ then by (11), (8),

$$d(B^{t-1} \overset{r}{\times} C_u, \omega[l, l + m_{t-1}|C_u| - 1]) < \frac{\delta^2}{3} \tag{83}$$

where $l = \widetilde{l} + |\widetilde{E_1}|$.

According to the Proposition 11

$$d(C_u, \omega_t[l', l' + |C_u| - 1]) < \frac{\delta^2}{3} \tag{84}$$

for some $l' \in \mathbb{Z}$ and

$$|l - p_t l'| < \frac{1}{3}\delta^2 r 2^{r+1} p_t. \tag{85}$$

We can write

$$\bar{C_u} = \bar{C_u}^{(1)} \bar{C_u}^{(2)} \bar{C_u}^{(3)}$$

according to (73).

We distinguish among the blocks $A_1, \ldots, A_{d_1}$ three types $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, as follows;

$$A_u \in \mathcal{F}_1 \text{ if } C_u \text{ is as in A) or G4)},$$

$$A_u \in \mathcal{F}_2 \text{ if } C_u \text{ is as in G1), G2), or G3)},$$

$$A_u \in \mathcal{F}_3 \text{ if } C_u \text{ is as in B)}.$$

Let $d_{11} = \#\mathcal{F}_1,\ d_{12} = \#\mathcal{F}_2,\ d_{13} = \#\mathcal{F}_3$. We have

$$d_{11} + d_{12} + d_{13} = d_1.$$

Let $\omega(A_1, \ldots, A_{d_1})$ be a subsequence of ω that is $\frac{\delta^2}{9}$-covered by the blocks $A_1, \ldots, A_{d_1}$ in a disjoint way. By $\omega(\mathcal{F}_i),\ i = 1, 2, 3$, we denote the subsequence of ω $\frac{\delta^2}{9}$-covered in a disjoint way by the families $\mathcal{F}_i$. Of course, $\omega(A_1, \ldots, A_{d_1}) \subset \omega(\mathcal{F}_1) \cup \omega(\mathcal{F}_2) \cup \omega(\mathcal{F}_3)$. Denoting by $\bar{\omega}(A_1, \ldots, A_{d_1}),\ \bar{\omega}(\mathcal{F}_i)$ the complementary parts of $\omega(A_1, \ldots, A_{d_1}),\ \omega(\mathcal{F}_i),\ i = 1, 2, 3$, respectively, we have

$$\bar{\omega}(A_1, \ldots, A_{d_1}) \supset \bar{\omega}(\mathcal{F}_1) \cap \bar{\omega}(\mathcal{F}_2) \cap \bar{\omega}(\mathcal{F}_3).$$

According to (83),(84),(85) and the Proposition 15 we have that $\bar{\omega}(\mathcal{F}_3)$ is an union of at least

$$(r - d_{13}) \text{ blocks } E(\delta_1), \tag{86}$$

where

$$\begin{cases} E = B^t \overset{r}{\times} b^{t+1}(i_j), \quad j = 1, \ldots, r - d_{13}, \text{ and} \\ \delta_1 \le 2\delta^2 r 2^{r+1}, \end{cases} \tag{87}$$

because of $\frac{|\widetilde{E}_1|}{|A_u|} \overset{(84)}{\le} \frac{p_t}{m_{t+1}} = \frac{r}{\lambda_{t+1}} < \frac{1}{2}\delta_1$, and $\frac{|\widetilde{E}_2|}{|A_u|} < \frac{1}{2}\delta_1$.

Consider the family $\mathcal{F}_2$. Let $A_u \in \mathcal{F}_2$. If A_u $\frac{\delta^2}{9}$-covers a fragment I_u of ω then (83) and (84) imply that $\bar{C}_u$ $\frac{\delta^2}{3}$-covers a fragment $I_u = I_u(t)$ of ω_{t+1} and (85) implies

$$I_u \subset (B^t \overset{r}{\times} I_u(t))(\delta_1).$$

It follows from the Proposition 14 that there is $A_{\bar{u}}$ of a form as in $\mathcal{F}_3$ such that $\widetilde{C}_{\bar{u}}$ $\frac{\delta^2}{3}$-covers another fragment $I_{\bar{u}}(t)$ of ω_{t+1} such that

$$I_u(t) \subset I_{\bar{u}}(t)(\delta).$$

Applying the Proposition 15 to the family $\{A_{\bar{u}}\}$ we obtain that $\bar{\omega}(\mathcal{F}_3) \cap \bar{\omega}(\mathcal{F}_2)$ is an union of at least $(r - d_{13} - d_{12})$ blocks $E(\delta_2)$, E is as (87) and $\delta_2 = \max(\delta, \delta_1)$.

Each block $E(\delta_2) \in \bar{\omega}(\mathcal{F}_3) \cap \bar{\omega}(\mathcal{F}_2)$ is an union of at least $(r - d_{13} - d_{12})$ blocks of the form $B^t \overset{r}{\times} e_k^{(s)},\ k \in \mathbb{Z}_n,\ s \in S,\ \#S = r - d_{13} - d_{12}$.
Using the same arguments as before we get that

$$\begin{cases} \bar{\omega}(\mathcal{F}_3) \cap \bar{\omega}(\mathcal{F}_2) \cap \bar{\omega}(\mathcal{F}_1) \text{ is an union} \\ \text{of at least } (r - d_{13} - d_{12} - d_{11}) \text{ blocks of the form } B^{t-1} \overset{r}{\times} e_k^{(s)}, \\ s \in S_1,\ \#S_1 = r - d_{13} - d_{12} - d_{11}. \end{cases} \tag{88}$$

Denoting $P(\omega_1, \omega)$ the density of a subsequence ω_1 in ω and using (69),(86), (88) we have
$P(\bar{\omega}(A_1, \ldots, A_{d_1}), \omega) \ge P(\bar{\omega}(\mathcal{F}_3) \cap \bar{\omega}(\mathcal{F}_2) \cap \bar{\omega}(\mathcal{F}_1), \omega)$

$\geq (1 - \frac{d_{13}+d_{12}}{r})(1 - \frac{d_{11}}{r})(\frac{1}{4nr})^2(1 - \delta_2)$

$\geq (1 - \frac{1}{r})^2(\frac{1}{4nr})^2(1 - \delta_2) \geq (1 - \frac{1}{r})^2(\frac{1}{4nr})^2\frac{1}{2}$.

If $T_1 \neq \{1, \ldots, x\}$ then we repeat the above reasoning to the subsequence $\bar{\omega}(\mathcal{F}_3) \cap \bar{\omega}(\mathcal{F}_2) \cap \bar{\omega}(\mathcal{F}_1)$ and $t = \tau_2$, and so on. As a consequence we get

$$P(\bar{\omega}(A_1, \ldots, A_x), \omega) \geq (1 - \frac{1}{r})^{2r}\frac{1}{2^r}(\frac{1}{4nr})^{2r}.$$

This implies $r(T_\varphi) > r - 1$. Thus we have shown $r(T_\varphi) = r$. ■

4.4 $r(T_\varphi) = r$: the case 3.1.

To prove that $r(T_\varphi) = r$ in the case 3.1 we can repeat the same arguments an in 4.3. Similarly as in the Theorem 4 we consider blocks A_u, $u = 1, \ldots, x$, $x \leq r - 1$, and A_u are as in (82), $C_u = b^t \overset{r}{\times} \bar{C}_u$ but $\bar{C}_u$ are as in A), B') and G1), G2), G3), G4).

As an analogue of the Propositions 13-15 and Theorem 4 we obtain

Proposition 13′ *Assume that C is as in B') and let $d(C, D) < \delta^2, D \subset \omega_t$. Then D has a form (75), and $\bar{D} =$
$b_{k'_{i_0}}(i_0)..b_{k'_{r-1}}(r-1) \mid b_{u'_0}(0)..b_{u'_{r-1}}(r-1) \mid b'_{v_0}(0)..b_{v'_{r-1}}(r-1) \mid b_{l'_0}(0)..b_{l'_{i_1}}(i_1)$,
$b_k(i) = b_k^{t+1}(i)$, and j satisfies either (76) or (77).*

Proposition 14′ *Let C be as in the Proposition 14, $\bar{C}$ is as in (73) and $\bar{C}_2$ is as in B'). Then we get g1),g2), g3) or g4).*

The proofs of the Propositions 13' and 14' are similar to the proofs of the Propositions 13 and 14.

Proposition 15′ *Let $\mathcal{F} = \{C_1, \ldots, C_d\}, d \leq r - 1, C_j = b^t \overset{r}{\times} \bar{C}$, and C_j are as in B'). Then we have the same thesis as in the Proposition 15.*

Proof. Let $\mathcal{F}_{i,k}$, $i = 0, \ldots, r - 2$, $k \in \mathbb{Z}_n$, be the set of all blocks $C \in \mathcal{F}$ such that $\bar{C}$ ($C = b^t \overset{r}{\times} \bar{C}$) δ^2-covers a subblock of ω_{t+1} containing one of the form $b_k^{t+1}(i)b_{k+1}^{t+1}(i+1)$. By $\mathcal{F}^{(1)}_{r-1,k}, \mathcal{F}^{(2)}_{r-1,k}$ we denote those $C \in \mathcal{F}$ such that $\bar{C}$ does so for the pairs $b_k^{t+1}(r-1)b_k^{t+1}(0)$ or $b_k^{t+1}(r-1)b_{k+1}^{t+1}(0)$ respectively.

Using the Propositions 3 and 7 we get that

$$(89) \quad \begin{cases} \text{if } C \in \mathcal{F}_{i,k} \text{ then } \bar{C}\ \delta^2\text{-covers (up to } \delta r^2 2^{r+3}\lambda_{t+1}) \text{ only those} \\ \text{fragments of } \omega_{t+1} \text{ containing blocks of the form} \\ (89') \qquad b^{t+1} \overset{r}{\times} \bar{e}_k(i+1) \text{ or } b^{t+1} \overset{r}{\times} \bar{e}_{k+4}(i), \text{ if } n \geq 3, \\ \textit{and} \\ (89'') \qquad b^{t+1} \overset{r}{\times} \bar{e}_k(i+1) \text{ if } n = 2, \end{cases}$$

whenever $i = 0, \ldots, r - 2$, $k \in \mathbb{Z}_n$. Using the Propositions 4 and 7 we get that

$$(90) \quad \begin{cases} \text{if } C \in \mathcal{F}^{(1)}_{r-1,k} \text{ then } \bar{C}\ \delta^2\text{-covers only those fragments} \\ \text{of } \omega_{t+1} \text{ containing blocks of the form} \\ (90') \qquad b_k^{t+1}(r-1)b_k^{t+1}(0) \text{ or } b_l^{t+1}(r-1)b_{l+1}^{t+1}(0), \\ l \text{ satisfies (54)}, \end{cases}$$

and

$$(91)\qquad \begin{cases} \text{if } C \in \mathcal{F}^{(2)}_{r-1,k} \text{ then } \bar{C}\ \delta^2\text{-covers only those fragments} \\ \text{of } \omega_{t+1} \text{ containing blocks of the form} \\ (91') \qquad b^{t+1}_k(r-1)b^{t+1}_{k+1}(0) \text{ or } b^{t+1}_l(r-1)b^{t+1}_l(0), \\ l \text{ satisfies (54).} \end{cases}$$

Now notice that each two blocks $b^{t+1} \overset{r}{\times} e^{(i)}_k$ and $b^{t+1} \overset{r}{\times} e^{(i)}_{k'},\ k' \in \mathbb{Z}_n,\ k \neq k'$, appearing in ω_{t+1} are separated by at least three blocks of the form $b^{t+1} \overset{r}{\times} e_{k+1}$. This, (89) and the condition $|E| < 4r$ (see B')) imply that $\mathcal{F}_{i,k} \cap \mathcal{F}_{i,k'} = \emptyset$, if $k \neq k',\ i = 0, \ldots, r-2$. Similarly $\mathcal{F}^{(1)}_{r-1,k} \cap \mathcal{F}^{(1)}_{r-1,k'} = \emptyset$ and $\mathcal{F}^{(2)}_{r-1,k} \cap \mathcal{F}^{(2)}_{r-1,k'} = \emptyset$, if $k \neq k'$.

Further (89) implies that if $C \in \mathcal{F}_{i,k} \cap \mathcal{F}_{i',k'}$ then $i' = i+1,\ k' = k+4$ if $n \geq 3$ ((89')) and $i' = i,\ k' = k$ if $n = 2$ ((89")), $i = 0, \ldots, r-2$. (90) implies that if $C \in \mathcal{F}^{(1)}_{r-1,k} \cap \mathcal{F}^{(2)}_{r-1,k'}$ then $k' = l$, l satisfying (54). Combining the above arguments we get that there is at least $\frac{rn}{2} - d$ fragments of ω_{t+1} of the form (89') and (90) or (91) that are not covered by the family $\mathcal{F}$. The Proposition follows because $\frac{rn}{2} \geq r$. ■

Theorem 4′ $r(T_\varphi) = r$.

Proof. We repeat the same reasoning as in the proof of the Theorem 4 using blocks $A_1, \ldots, A_x$ of the form (82) with $q \geq 3$. We use the Proposition 7 instead of the Proposition 11 and the Propositions 14′ and 15′ instead of the Propositions 14 and 15. The using (67) instead of (68) we get

$$P(\bar{\omega}(A_1, \ldots, A_x), \omega) \geq (1 - \frac{1}{r})^{2r} \frac{1}{2^r} (\frac{1}{rn2^{r+1}})^{2r},$$

what implies $r(T_\varphi) > r-1$ and by Proposition 12 we have $r(T_\varphi) = r$. ■

5 Pairs (r, ∞) or (∞, m).

In this part we construct group extensions $(X \times G, T_\varphi)$ such that $r(T_\varphi) = r,\ \ q(T_\varphi) = \infty,\ 2 \leq r < \infty$ or $r(T_\varphi) = \infty,\ \ q(T_\varphi) = m,\ 1 \leq m < \infty$.

5.1 The case (r, ∞).

Take a sequence $\{s_t\}_{t=0}^\infty,\ s_{t+1} = \mu_{t+1}s_t,\ s_0 = \mu_0,\ \mu_t \geq 2$ for $t \geq 0$ and let G be the group of $\{s_t\}$-adic integers. Let $e = 1 + 0s_1 + 0p_2 + ...$ The set of all $\{s_t\}$-adic rational integers of G coïncides with the set $\{e_n, n \in Z\}$, where $e_n = ne$. Similarly as in the case 3.1 we define an adding machine $(X, \mathcal{B}, \mu, T)$ and a cocycle $\varphi : X \longrightarrow G$. To do this we define blocks $F^{(0)}, F^{(1)}, ..., F^{(r-1)}$ ($r \geq 2$ is given) over G.
Put

$$F^{(i)}(t) = F^{(i)} = \overbrace{0\ldots0}^{r(2^{i+1}-1)}\ \overbrace{0\ldots0e0\ldots0}^{r},\ \ i = 0, \ldots, r-1$$

$H^{(i)} = F^{(i)}F^{(i)}_e \ldots F^{(i)}_{(s_t-1)e}$.
Then $|H^{(i)}| = s_t r 2^{t+1}$. Next define $b^t(0), \ldots, b^t(r-1)$ as in 3.1 and
$b^t = b^t(0) \ldots b^t(r-1),\ t \geq 0$.
We have

$$\lambda_t = |b^t(i)| = s_t r 2^{r+t+1},\ i = 0, \ldots, r-1$$

and

$$|b^t| = s_t r^2 2^{r+t+1}.$$

Then we define the blocks $B^t,\ t \geq 0$ by (14). We have $p_t = |B^t| = s_0 \dots s_t r^{2t} 2^{r+1}(2^{t+1} - 1)$. Let $(X, \mathcal{B}, \mu, T)$ be the $\{p_t\}$-adic adding machine and define a cocycle $\varphi : X \longrightarrow G$ by (18).

Theorem 5 *$r(T\varphi) = r$ and $q(T_\varphi) = \infty$*

Proof. Let $\Pi_t : G \longrightarrow Z/s_tZ$ be the natural group homomophism. We can define cocycles $\varphi_t : X \longrightarrow Z/s_tZ$ by $\varphi_t = \varphi \circ \Pi_t$. It is evident that φ_t is a r-Toeplitz cocycle as in 3.1 defined by the blocks $\Pi_t(B_k), u \geq 0$. According to Theorems 2 and 4 we have $r(T_{\varphi_t}) = r$ and $q(T_{\varphi_t}) = s_t$. It follows from the definitions of φ and φ_t that the dynamical system $(X \times G, T_\varphi)$ is the inverse limit of the systems $(X \times Z/s_tZ, T_{\varphi_t})$. Then from the definition of the rank we obtain $r(T_\varphi) = r$. It is proved in Theorem 2 that $\sigma_{je} \notin wcl\{T^n_{\varphi_t}, n \in Z\}$ if $j = 0, \dots, s_t - 1,\ t \geq 0$. This means that $\sigma_{je} \notin wcl\{T^n_\varphi, n \in Z\}$ for every $j \in Z,\ j \neq 0$ which implies $q(T_\varphi) = \infty$. ■

5.2 The case (∞, m).

First consider the case $m \geq 2$. Let $r_t = 2^{t+1},\ t \geq 0$ and define blocks $F^{(i)} = F^{(i)}(t)$ over $G = Z/mZ,\ i = 0, \dots, r_{r+1} - 1$ as follows:

$$F^{(i)} = \overbrace{0\dots0}^{2^{i+1}r_t}\overbrace{0\dots0\underbrace{\ 1\ }_{i+1}0\dots0}^{r_{t+1}},$$

$$H^{(i)} = F_0^{(i)} F_1^{(i)} \dots F_{m-1}^{(i)} \quad i = 0, \dots, r_{t+1} - 1.$$

We have $|H^{(i)}| = mr_t 2^{i+3}$. Next define $b^t(0), \dots, b^t(r_{t+1} - 1), b^t, B^t$ by putting

$$b^t(i) = \overbrace{H^{(i)} H^{(i)} \dots H^{(i)}}^{x}, x = 2^{t+r_{t+1}-i-1}$$

$$b^t = b^t(0) b^t(1) \dots b^t(r_{t+1} - 1), \quad \text{and}$$

$$B^t = b^0 \overset{r_0}{\times} b^1 \overset{r_1}{\times} \dots \overset{r_{i-1}}{\times} b^t.$$

Then $\lambda_t = |b^t(i)| = m 2^{2t+\rho+2}, \rho = r_{t+1}$ and $p_t = m_t r_{t+1},\ m_t = \lambda_0 \dots \lambda_t$. We define a cocycle $\varphi : X \longrightarrow G$ by

$$\varphi(x) = B^t[j+1] - b^t[j]$$

if $x \in D^t_j$ except if $j = m_t - 1, \dots, p_t - 1$. The cocycle φ is constant on the levels D^t_j except of r_{t+1} consecutive levels.

In a similar way we construct a cocycle φ if $m = 1$. Take n as in the case 3.2 and define

$$F^{(i)}(t) = F^{(i)} = \overbrace{0\dots0}^{2r_t}\overbrace{0\dots0\underbrace{\ 1\ }_{i+1}0\dots0}^{2r_t}$$

$$H^{(i)} = F_0^{(i)} F_1^{(i)} \dots F_{n-1}^{(i)}, \quad i = 0, 1, \dots, r_{t+1} - 1.$$

The next steps of the definition φ are the same as in the case $m \geq 2$.

Theorem 6 *$r(T_\varphi) = \infty, q(T_\varphi) = m$ and $wcl\{T_\varphi^n, n \in z\}$ is uncountable.*

Proof. For the dynamical system $(X \times G, T_\varphi)$ we can use the same arguments as in the parts 3 and 4 taking r_t instead of r. The Theorems 2,3 and $3'$ are valid. To estimate the rank of T_φ we use the shift representations $(\Omega\omega, T_\sigma)$ of $(X \times G, T_\varphi)$ where $\omega = b^0 \overset{r_0}{\times} b^1 \overset{r_1}{\times} \dots$. Repeating the proof of the Theorem 4 and $4'$ we get $r(T_\varphi) > r_t - 1$ for every $t \geq 0$. Thus $r(T_\varphi) = \infty$. ■

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