RENORMALIZATION OF ALGORITHMS IN THE PROBABILISTIC SENSE

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Dedicated to Professor J. Kubilius on the occasion of his 75^{th} birthday.

Abstract. For a sequence $(\mathcal{P}_n)$ of measurable partitions of a continuous probability space $(\Omega, \mathcal{B}, \mu)$, which refines to point partition $\mathcal{E}$, and given $(\Gamma, \mathcal{C}, \nu)$ such another space, we introduce a renormalization process. It is a sequence $(t_n : \Omega \to \Gamma)$ of random variables such that for $A \in \mathcal{C}$, if $\mathcal{P}_n = \{P_{n,i} : i \in \Delta_n\}$, then $\mu(P_{n,i} \cap t_n^{-1}(A)) \,/\, \mu(P_{n,i}) = \nu(A)$. We derive criterions for that μ-a.s., $\frac{1}{N}\sum_{n=0}^{N-1} \mathbf{1}_A(t_n(x)) \to \nu(A)$, which also give a speed of convergence when applicable. Several examples or counterexamples illustrate the phenomenon and its analysis. Links to "Cantor-Bernstein" type lemmas, a.s. convergence in L^p spaces, and central limit theorems are also indicated.

1. Introduction

The *renormalization* of number systems, though not clearly conceptualised, was studied in [Sc, Chap. 11], [La] and [LaTh] for number systems of the interval $[0,1]$. The aim of this paper is to build a general formalism for it, and the related notion of *adapted barycentrical positions.* We shall also include some new directions of application of these concepts, namely to "Cantor-Bernstein" type lemmas, central limit theorems or a.s. convergence in L^p spaces. So let us start with some notations and definitions.

Both $(\Omega, \mathcal{B}, \mu)$ and $(\Gamma, \mathcal{C}, \nu)$ shall denote continuous probability spaces. By an *algorithm* on $(\Omega, \mathcal{B}, \mu)$ we mean a quadruple $\mathcal{T} = (\Omega, \mathcal{B}, \mu, \mathcal{P})$ where $\mathcal{P} = (\mathcal{P}_n)$ is a refining sequence of measurable partitions of Ω, such that $\bigvee_n \mathcal{P}_n = \mathcal{E}$, the point partition. If $\mathcal{P}_n = \{P_{n,i} : i \in \Delta_n\}$, we require each $P_{n,i}$ to have positive measure. Such algorithms may be thought of as number systems in the classical sense (dyadic or continued fraction expansions, etc, with interval partitions) (cf. [Sc] for various examples).

Given such $\mathcal{T}$, μ-a.s., for $x \in \Omega$, there exists a unique sequence $(i_n(x)) \in \prod \Delta_n$ such that $x \in P_{n,i_n(x)}$, for all n. The sequence $(i_n(x))$ is the *digital expansion of x.*

The second main objects we shall deal with are, associated to given $\mathcal{T}$ and $(\Gamma, \mathcal{C}, \nu)$, the sequences of *adapted barycentrical positions.* These are sequences $(t_n : \Omega \to \Gamma)$ of a.s. onto random variables satisfying, for each n,

$$\mu(P_{n,i} \cap t_n^{-1}(A)) \,/\, \mu(P_{n,i}) = \nu(A),\ A \in \mathcal{C},\ i \in \Delta_n. \tag{1.1}$$

The set of such sequences, given $\mathcal{T}$, shall be denoted by $ABP(\mathcal{T})$.

When both $(\Omega, \mathcal{B}, \mu)$ and $(\Gamma, \mathcal{C}, \nu)$ are continuous Lebesgue probability spaces [Ro] then obviously $ABP(\mathcal{T}) \neq \emptyset$.

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Renormalizing the algorithm $\mathcal{T}$ consists in choosing a "standard screen" $(\Gamma, \mathcal{C}, \nu)$ and a sequence (t_n) in $ABP(\mathcal{T})$. Then given a point x in Ω, we obtain a sequence $(t_n(x))$ in Γ (see Figure 1); the purpose of this paper is the investigation of its "uniform distribution" in Γ; more precisely, given $A \in \mathcal{C}$, is it true or not that

$$\mu - a.e., \quad \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{1}_A(t_n(x)) \to \nu(A)? \tag{1.2}$$

This question is not completely new, though it was never stated but in the restricted context of number systems of the interval [LaTh], [Sc, Chap. 11]. The paper is organized as follows.

In Section 2, we present the general preliminary lemmas or definitions, and obtain in Theorem 2.1 the key tool we shall use in latter sections when dealing with specified algorithms. In Section 3 we extend some previous results for number systems of the interval. Then Theorem 3.2. and Counter-example 3.1. illustrate how far Eq. (1.2) is true in general. We also point out some elementary connections with dynamical systems in Remark 3.1 [Pa], [Wa].

In Section 4 we show what may happen and be computed when renormalization is processed on number systems of $[0,1]^2$. And Section 5 gives some hints towards obtaining via renormalizations "Cantor-Bernstein" lemmas, central limit theorems or a.s. convergence along adapted barycentrical positions in L^p spaces.

2. Preliminary results and notations

Hereafter, inclusions or equalities among sets are understood mod 0. Let $(t_n : \Omega \to \Gamma) \in ABP(\mathcal{T})$. Then whenever $A \in \mathcal{C}$, Eq. (1.1) obviously implies that

$$\mu(t_n^{-1}(A)) = \nu(A). \tag{2.1}$$

Definition 2.1. *For $m \geq 0$, and $E \in \mathcal{B}$, let $\mathcal{P}_m(E) = \{P_{m,i} \in \mathcal{P}_m : P_{m,i} \subset E\}$, and $\mathcal{P}_m(\partial E) = \{P_{m,i} \in \mathcal{P}_m : \mu(P_{m,i} \cap E)\mu(P_{m,i} \cap E^c) \neq 0\}$. Then put*

$$\begin{cases} (i) \quad Int_m(E) = \bigcup_{\mathcal{P}_m(E)} P_{m,j} \,, \\ (ii) \quad \partial_m(E) = \bigcup_{\mathcal{P}_m(\partial E)} P_{m,i}. \end{cases}$$

Chosen $A \in \mathcal{C}$, let $E_n = t_n^{-1}(A)$, $n \geq 0$. Then for $m \geq 1$, it follows from Definition 2.1 that

$$E_n \cap E_{n+m} = (Int_{n+m}(E_n) \cap E_{n+m}) \cup (\partial_{n+m}(E_n) \cap E_n \cap E_{n+m}).$$

With Eqs. (2.1) and (1.1) we obtain

$$\begin{cases} \mu(Int_{n+m}(E_n) \cap E_{n+m}) = \nu(A)\mu(Int_{n+m}(E_n)), \\ \\ \mu(\partial_{n+m}(E_n) \cap E_n \cap E_{n+m}) \leq \mu(\partial_{n+m}(E_n) \cap E_{n+m}) = \nu(A)\mu(\partial_{n+m}(E_n)), \end{cases}$$

It follows that $\mu(E_n \cap E_{n+m}) \leq \nu(A)\left(\mu(Int_{n+m}(E_n)) + \mu(\partial_{n+m}(E_n))\right)$. Since $\mu(E_n) = \mu(E_{n+m}) = \nu(A)$, $Int_{n+m}(E_n) \subseteq Int_{n+m+1}(E_n) \subseteq E_n$, and $E_n \setminus Int_{n+m}(E_n) \subseteq \partial_{n+m+1}(E_n) \subseteq \partial_{n+m}(E_n)$, we see that if

$$\alpha_{n,m} = \nu(A)(\mu(\partial_{n+m}(E_n)) - \mu(E_n \setminus Int_{n+m}(E_n))),$$

then $\alpha_{n,m} \geq \alpha_{n,m+1} \geq 0$ and also $\alpha_{n,m} \leq \nu(A)\mu(\partial_{n+m}(E_n))$. Hence we have proved the *mixing relation*

Lemma 2.1. *For $n \geq 0$, $m \geq 1$, $\mu(E_n \cap E_{n+m}) - \mu(E_n)\mu(E_{n+m}) \leq \alpha_{n,m}$, and*

$$\begin{cases} (i) \quad \alpha_{n,m} \leq \nu(A)\mu(\partial_{n+m}(E_n)), \\ \\ (ii) \quad \alpha_{n,m} \geq \alpha_{n,m+1} \geq 0. \end{cases}$$

Theorem 2.1. *Define, for integers $0 \leq p < q$, $r \geq 1$, $0 \leq s \leq r$, and t,*

$$\begin{cases} (i) & L_r(s) = \{I =]t2^s, (t+1)2^s] : I \subseteq]0, 2^r]\}; \\ (ii) & L_r = \cup_{0 \leq s \leq r} L_r(s); \\ (iii) & \gamma(p,q) = \sum_{p<i\leq q} \left(\sum_{0\leq k\leq q-i} \alpha_{i,k}\right); \\ (iv) & \Psi(r) = \displaystyle\sum_{]u,v]\in L_r} \gamma(u,v). \end{cases}$$

Then for any $\varepsilon > 0$, μ-a.s., if $2^{r-1} < N \leq 2^r$,

$$\left| \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{E_i}(x) - \nu(A) \right| = \mathcal{O}(r^{1+\varepsilon} \frac{\sqrt{\Psi(r)}}{2^r}).$$

Proof. Let us define $A(p,q,x) = \sum_{p<i\leq q} \mathbf{1}_{E_i}(x)$; $A(q,x) = A(0,q,x)$, $\phi(p,q) = \nu(A)(q-p)$; $\phi(q) = \phi(0,q)$ $(= q\nu(A))$. Then since $\phi(p,q) = \int_\Omega A(p,q,x)d\mu(x)$, using (iii), we may compute, with Lemma 2.1,

$$\int_\Omega (A(p,q,x) - \phi(p,q))^2 d\mu(x) = \sum_{p<i,j\leq q} \mu(E_i \cap E_j) - \mu(E_i)\mu(E_j)$$

$$\leq 2\left(\sum_{p<i\leq j\leq q} \mu(E_i \cap E_j) - \mu(E_i)\mu(E_j)\right) \leq 2\gamma(p,q).$$

Let $Z_r(x) = \sum_{]u,v]\in L_r} (A(u,v,x) - \phi(u,v))^2$ (cf. (ii)). With (iv) and the last inequality above we see that $\int_\Omega Z_r(x)d\mu(x) \leq 2\Psi(r)$. Hence if $\varepsilon > 0$,

$$\sum_{r\geq 1} \int_\Omega Z_r(x) \,/\, (r^{1+\varepsilon}\Psi(r))d\mu(x) < \infty,$$

from which it follows by the monotone convergence theorem that μ-a.s., $Z_r(x) = \mathcal{O}(r^{1+\varepsilon}\Psi(r))$.

Let N be such that $2^{r-1} < N \leq 2^r$; then $]0, N]$ can be partitioned with at most $r+1$ intervals from L_r ((i), (ii)); call $L_r(N)$ the corresponding family; $\#L_r(N) \leq r+1$. Since $A(N,x) - \phi(N) = \sum_{]u,v]\in L_r(N)} A(u,v,x) - \phi(u,v)$, using Cauchy's inequality $((x_1 + \ldots x_n)^2 \leq n(x_1^2 + \ldots x_n^2))$, we obtain $(A(N,x) - \phi(N))^2 = \mathcal{O}(rZ_r(x)) = \mathcal{O}(r^{2+\varepsilon}\Psi(r))$. The conclusion follows. □

The first corollary shows that under particular assumptions on the $\alpha_{n,m}$'s, we find the same estimation in Theorem 2.1 as the one appearing in [Ph]:

Corollary 2.1. *If $\alpha_{n,m} \le c_m$ and $\sum c_m = S < \infty$, then*

$$\left|\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{E_i}(x) - \nu(A)\right| = \mathcal{O}(\frac{\log^{\frac{3}{2}+\varepsilon} N}{\sqrt{N}}).$$

Proof. With the assumptions made on the $\alpha_{n,m}$'s, we have $\gamma(0,2^r) \le S2^r$, and since $\Psi(r) \le (r+1)\gamma(0,2^r)$, when $2^{r-1} < N \le 2^r$, we obviously have $r^{1+\varepsilon}\frac{\sqrt{\Psi(r)}}{2^r} = \mathcal{O}(\frac{\log^{\frac{3}{2}+\varepsilon} N}{\sqrt{N}})$. □

The second corollary shows that the $\alpha_{n,m}$'s may go rather slowly to 0 and still $r^{1+\varepsilon}\frac{\sqrt{\Psi(r)}}{2^r} \to 0$ with $r \to \infty$:

Corollary 2.2. *If $\alpha_{n,m} \le \beta_m = o\left(\frac{1}{\log^{3+\varepsilon}(m)}\right)$, then*

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{E_i}(x) \xrightarrow[N\to\infty]{} \nu(A)\ \mu - a.s..$$

Proof. It can be shown, using Bertrand's integrals and similar estimations as in the proof of Corollary 2.1, that as $r \to \infty$, the assumptions on the $\alpha_{n,m}$'s imply $r^{1+\varepsilon}\frac{\sqrt{\Psi(r)}}{2^r} \to 0$. □

3. Renormalization of number systems of the interval

For number systems on $[0,1]$ [LaTh] we let, if not specified, $(\Omega,\mathcal{B},\mu) = ([0,1],\mathcal{B},m)$ where m is Lebesgue measure and $\mathcal{B}$ the associated Borel σ-algebra. We also assume any $P_{n,i}$ to be a subinterval, not specifying inclusion or exclusion of the associated endpoints; this way set $P_{n,i} = (a_{n,i}, b_{n,i})$. In [LaTh] the m-a.s. uniform or complete uniform distributions mod 1 [KuNi] of the sequence $(t_n(x)) = \left(\frac{x-a_{n,i_n(x)}}{b_{n,i_n(x)}-a_{n,i_n(x)}}\right)$ is studied. The following result, proved as in [LaTh], using Corollary 2.1., strengthens slightly [LaTh, Thm 3.1.];

Theorem 3.1. *Let $t_n(x) = \frac{x-a_{n,i_n(x)}}{b_{n,i_n(x)}-a_{n,i_n(x)}}$ or $t_n(x) = \frac{b_{n,i_n(x)}-x}{b_{n,i_n(x)}-a_{n,i_n(x)}}$, $n \ge 0$. If there is a $q < 1$ such that whenever $(a_{n+1,j}, b_{n+1,j}) \subseteq (a_{n,i}, b_{n,i})$, $(b_{n+1,j} - a_{n+1,j}) \,/\, (b_{n,i} - a_{n,i}) < q$, then m-a.s., for any $d \in]0,1]$, $\varepsilon > 0$,*

$$\left|\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{[0,d)}(t_n(x)) - d\right| = \mathcal{O}(\frac{\log^{\frac{3}{2}+\varepsilon} N}{\sqrt{N}}).$$

Let us now point out that if $\alpha_{n,m} \ge c > 0$, then $\Psi(r) \ge 2^{2r}$, and hence $r^{1+\varepsilon}\frac{\sqrt{\Psi(r)}}{2^r}$ does not go to 0 as $r \to \infty$. Therefore the question of weither $r^{1+\varepsilon}\frac{\sqrt{\Psi(r)}}{2^r} \to 0$ for any of the renormalization processes we define arises. This question may be settled as follows; can mixing hold (i.e. $\alpha_{n,m} \to_{m\to\infty} 0$) while "uniform distribution" (cf. (1.2)) does not?

We shall answer this question by two steps; first in Theorem 3.2. give general conditions for mixing to hold, and next in Counter-example 3.1. give a renormalization for which those conditions hold while Eq. (1.2) does not.

Theorem 3.2. *Suppose Ω is a topological space and $\mathcal{B}$ is its Borel σ-algebra (eventually μ-completed). Assume the following conditions hold (when $E \subseteq \Omega$, $Fr(E)$ denotes its topological boundary):*

$$\begin{cases} (i) & \mu(O) > 0 \text{ if } O \text{ is open and non-empty;} \\ (ii) & \mu(Fr(P_{n,i})) = 0,\ i \in \Delta_n,\ n \ge 0; \\ (iii) & V(\mathcal{P}) := \{\cup_{j\in J} P_{n,j} : \emptyset \ne J \subseteq \Delta_n,\ n \ge 0\} \\ & \text{is a basis of neighbourhoods for all points.} \end{cases}$$

Then if $E \in \mathcal{B}$,

$$(\mu(Int_m(E)) \to \mu(E) \text{ and } \mu(\partial_m(E)) \to 0) \Longleftrightarrow (\mu(Fr(E)) = 0)\,.$$

Thus, if $\mu(Fr(E_n)) = 0$, $n \geq 1$, *then* $\alpha_{n,m} \to 0$ *as* $m \to +\infty$, *decreasingly, for each* n.

Proof. The last statement shall obviously follow, via Lemma 2.1., from the one preceding it. We shall briefly sketch the proof of the equivalence. If $C \subseteq \Omega$, let C° denote its topological interior.

To prove the implication $\Rightarrow$, first observe that from (i), (ii), and (iii) it follows that $E^\circ = \cup_{m\geq 1} Int_m(E)^\circ$. Since $(Int_m(E)^\circ)_{m\geq 1}$ increases (cf. proof of Lemma 2.1.) and $\mu \geq 0$, (ii) gives $\mu(\cup_{m\geq 1} Int_m(E)) = \mu(\cup_{m\geq 1}(Int_m(E))^\circ) = \lim_{m\to+\infty}\uparrow \mu(Int_m(E)^\circ) = \mu(E^\circ)$. Observe that $\cap_{m\geq 1}\partial_m(E) = \Omega \setminus (\cup_{m\geq 1}(Int_m(E) \cup Int_m(E^c)))$. Since $\mu(\partial_m(E)) \to 0$, and $\mu(\Omega) = 1$, the preceding applied to both E and E^c gives $0 = 1 - \mu(\cup_{m\geq 1} Int_m(E)) - \mu(\cup_{m\geq 1} Int_m(E^c)) = 1 - \mu(E^\circ) - \mu((E^c)^\circ) = \mu(Fr(E))$.

To prove the reverse implication $\Leftarrow$, take $x \in \cap_{m\geq 1}\partial_m(E)^\circ$. Let O be open and $x \in O$. Then from (i), (ii), (iii), and Definition 2.1, we get $\mu(E \cap O)\mu(E^c \cap O) > 0$. Hence O intersects E and E^c; therefore $\cap_{m\geq 1}\partial_m(E)^\circ \subseteq Fr(E)$. But from (ii), the hypothesis and the proof of Lemma 2.1., it follows that $0 = \mu(\cap_{m\geq 1}\partial_m(E)) = \lim_{m\to+\infty}\downarrow \mu(\partial_m(E))$. Finally as $E \subseteq Int_m(E) \cup \partial_m(E)$, and $Int_m(E) \subseteq E$, this implies $\lim \mu(Int_m(E)) \leq \mu(E) \leq \lim \mu(Int_m(E)) + \lim \mu(\partial_m(E)) = \lim \mu(Int_m(E))$. □

As is easily seen, the conditions of Theorem 3.1. are satisfied for number systems on the interval as defined at the beginning of this Section. Thus for these $\alpha_{n,m} \to_{m\to\infty} 0$ for any n.

Counter-example 3.1. *Let* $(\Omega, \mathcal{B}, \mu) = (\Gamma, \mathcal{C}, \nu) = ([0,1[, \mathcal{B}, m)$. *Let* $\mathcal{P}_n = \{[\frac{k}{2^n}, \frac{k+1}{2^n}[: 0 \leq k < 2^n\}$, $n \geq 1$ *(here* $\Delta_n = \{0, 1, \ldots, 2^n - 1\}$*). Let* $t_n(x) = 2^n x \bmod 1$. *Then* $(t_n) \in ABP(\mathcal{T})$. *Next choose* $0 = n_0 < n_1 < \ldots < n_k < n_{k+1} < \ldots$ *a subsequence of integers. Then define* $\mathcal{P}'_n = \mathcal{P}_{n_k}$ *whenever* $n_k \leq n < n_{k+1}$. *It is obvious that* $\mathcal{T}' = ([0,1[, \mathcal{B}, m, (\mathcal{P}'_n))$ *is an algorithm; if we put* $\bar{t}_n = t_{n_k}$ *when* $n_k \leq n < n_{k+1}$, *then* $(\bar{t}_n) \in ABP(\mathcal{T}')$. *Now let us suppose additionally that* $n_{k+1} \;/\; n_k \to \infty$. *Then for any* $d \in]0,1[$, *and any* $x \in [0,1[$, *the set of limit points of* $(\frac{1}{N}\sum_1^N \mathbf{1}_{[0,d)}(\bar{t}_n(x)))_{N\geq 1}$ *contains* 0 *or* 1. *Hence* (1.2) *does not hold.* □

In the two following remarks, we point out some occurences of renormalization and barycentrical positions.

Remark 3.1. *The map* $f(x) = \frac{2}{\pi} Arcsin\sqrt{x}$ *determines a topological* $\mathcal{C}^1$ *conjugacy between the classical quadratic map* $Tx = 4x(1-x)$ [CoEc] *and the tent map* $Sx = 2x$ *if* $0 \leq x \leq \frac{1}{2}$ *and* $Sx = 2 - 2x$ *if* $\frac{1}{2} \leq x \leq 1$ *(see* [Wa] *for dynamical systems and conjugacy). Since* S *preserves Lebesgue measure on* $[0,1]$ *and* f *is differentiable then* T *admits an absolutely continuous invariant measure with density* $f'(x)dx$. *But* $(S^n) \in ABP(\mathcal{T})$, *where* $\mathcal{T}$ *is as in the Counter-example 3.1. A straightforward computation shows that if we set* $\bar{\mathcal{P}}_n = f^{-1}(\mathcal{P}_n)$, *then* $\bar{\mathcal{T}} = ([0,1], \mathcal{B}, f'dm, (\bar{\mathcal{P}}_n))$ *is an algorithm and* $(T^n) \in ABP(\bar{\mathcal{T}})$, *when* $\bar{\mathcal{T}}$ *is renormalized to* $(\Gamma, \mathcal{C}, \nu) = ([0,1], \mathcal{B}, f'dm)$. *The interested reader may consult* [Pa] *for further examples in this direction.* □

Remark 3.2. *Saying* m*-a.s.,* $x \in [0,1[$ *is normal to base* 2 [Bo], [KuNi], *or that Eq.* (1.2) *is valid for the algorithm* $\mathcal{T}$ *of Counter-example 3.1., is strictly the same.*

4. Examples of renormalization when $\Omega = \Gamma = [0,1]^2$

For $(x,y) \in (0,1)^2$, let us write $x = \sum_{i\geq 0} \frac{x_i}{2^i}$ and $y = \sum_{j\geq 0} \frac{y_j}{2^j}$ denote there respective dyadic expansions ($x_i, y_j \in \{0,1\}$ and $x_0 = y_0 = 0$).

For $n \geq 0$ let $\mathcal{D}_n = \{C_n(k,l) := (\frac{k}{2^n}, \frac{k+1}{2^n}) \times (\frac{l}{2^n}, \frac{l+1}{2^n}) : 0 \leq k, l < 2^n\}$ be the partition of $(0,1)^2$ by so-called in the sequel *dyadic squares of order* n (the partition is one mod 0 with respect to Lebesgue measure m_2 on $(0,1)^2$). Let $\mathcal{B}$ be the Borel σ-algebra on $(0,1)^2$.

Example 4.1. Given $n \geq 0$ we shall slightly refine $\mathcal{D}_n$ to a new partition $\mathcal{P}_n$. Take some $C_n(k,l) \in \mathcal{D}_n$ and cut it into two rectangles of equal area according to the following rule (see Figure 2);

– if $k+l=0 \pmod 2$ then cut $C_n(k,l)$ by a parallel to the x axis in $\mathbb{R}^2$; suppose for instance $C_n(k,l)=(a,b)\times(c,d)$: then put $P_n(k,l,0)=(a,b)\times(c,\frac{c+d}{2})$ and $P_n(k,l,1)=(a,b)\times(\frac{c+d}{2},d)$;

– if $k+l=1 \pmod 2$ then cut it parallel to the y axis in $\mathbb{R}^2$: put $P_n(k,l,0)=(a,\frac{a+b}{2})\times(c,d)$ and $P_n(k,l,1)=(\frac{a+b}{2},b)\times(c,d)$.

Finally let $\mathcal{P}_n=\{P_n(k,l,\varepsilon): 0\le k,l<2^n, \varepsilon=0,1\}$. It is not difficult to check that indeed the sequence $(\mathcal{P}_n)_{n\ge 0}$ is refining and converging to $\mathcal{E}$. Thus an algorithm $\mathcal{T}_1=((0,1)^2,\mathcal{B},m_2,(\mathcal{P}_n)_{n\ge 0})$ is constructed. Next let us choose $(\Gamma,\mathcal{C},\nu)=((0,1)^2,\mathcal{B},m_2)$.

Lemma 4.1. *For $n\ge 0$, and $x,y\in(0,1)$ let*

$$t_n(x,y)=\left(2^{n+|x_n-y_n|}x \mod 1, 2^{n+1-|x_n-y_n|}y \mod 1\right).$$

Then $(t_n)\in ABP(\mathcal{T}_1)$.

Proof. If $(x,y)\in C_n(k,l)$ then $|x_n-y_n|=k+l \mod 2$. Assume for instance that $k+l=0 \mod 2$; let $d,d'\in]0,1]$. Then using the definition of $t_n(\cdot,\cdot)$ in Lemma 4.1, we get

$$\begin{cases} t_n^{-1}((0,d)\times(0,d'))\cap P_n(k,l,0)=(\frac{k}{2^n},\frac{k+d}{2^n})\times(\frac{l}{2^n},\frac{2l+d'}{2^{n+1}}); \\ t_n^{-1}((0,d)\times(0,d'))\cap P_n(k,l,1)=(\frac{k}{2^n},\frac{k+d}{2^n})\times(\frac{l}{2^n},\frac{2l+1+d'}{2^{n+1}}). \end{cases}$$

Similar formulas hold when $k+l=1 \mod 2$. The proof follows. □

In view of Theorem 2.1. we now proceed to the estimation of the $\alpha_{n,m}$'s. Let $V_n=t_n^{-1}((0,d)\times(0,1))$ and $H_n=t_n^{-1}((0,1)\times(0,d'))$, take $A=(0,d)\times(0,d')$ and let $E_n=t_n^{-1}(A)$, $n\ge 0$.

Since $E_n=V_n\cap H_n$, it follows that $\partial_{n+m}(E_n)\subseteq\partial_{n+m}(V_n)\cup\partial_{n+m}(H_n)$ (cf. Definition 2.1). By drawing a picture it appears obvious that $\max(m_2(\partial_{n+1}(V_n)),m_2(\partial_{n+1}(H_n)))\le\frac{3}{4}$, and inductively also obvious is the fact that

$$\max\left(m_2(\partial_{n+m}(V_n)),m_2(\partial_{n+m}(H_n))\right)\le\left(\frac{3}{4}\right)^m.$$

From $\partial_{n+m}(E_n)\subseteq\partial_{n+m}(V_n)\cup\partial_{n+m}(H_n)$ this leads to $\alpha_{n,m}\le 2\left(\frac{3}{4}\right)^m$, which in the light of Corollary 2.1 gives (see [KuNi] for uniformly distributed sequences in $(0,1)^2$)

Theorem 4.1. *For m_2 a.e. $(x,y)\in C$, and any $d,d'\in]0,1]$, any $\varepsilon>0$,*

$$\left|\frac{1}{N}\sum_{0\le n<N}\mathbf{1}_{(0,d)\times(0,d')}(t_n(x,y))-dd'\right|=\mathcal{O}\left(\frac{\log^{\frac{3}{2}+\varepsilon}N}{\sqrt{N}}\right).$$

Hence, m_2 a.e., the sequence $\left((2^{n+|x_n-y_n|}x \mod 1, 2^{n+1-|x_n-y_n|}y \mod 1)\right)_{n\ge 0}$ is uniformly distributed in $(0,1)^2$.

Example 4.2. Let us start again with dyadic squares of order n in $\mathcal{D}_n$ and refine them. Taking $C_n(k,l) \in \mathcal{D}_n$, let us cut it into 4 pieces, using the two diagonals of square $C_n(k,l)$. Let $\mathcal{P}_n$ be the partition deduced this way from $\mathcal{D}_n$. Then $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$, and $(\mathcal{P}_n)_{n\geq 0}$ converges to $\mathcal{E}$. Hence we have an algorithm $\mathcal{T}_2 = (C, \mathcal{B}, m_2, (\mathcal{P}_n))$. Let us choose again $(\Gamma, \mathcal{C}, \nu) = (C, \mathcal{B}, m_2)$.

Given $(x,y) \in C$ let $x(n) = \sum_{0\leq i\leq n} \frac{x_i}{2^i}$ and $y(n) = \sum_{0\leq j\leq n} \frac{y_j}{2^j}$, where we still refer to the dyadic expansions. Also put $x'(n) = x(n) + \frac{1}{2^n}$ and $y'(n) = y(n) + \frac{1}{2^n}$, $n \geq 0$.

If $M, N \in I\!R^2$, $M \neq N$, let $D(M,N)$ be the strait line passing both through points M and N. Figure 3 introduces some further notations; it pictures one of the four possibilities to locate X inside the nth order dyadic square containing it once this square has been cut through by its diagonals. However the construction to follow exhibiting some sequence in $ABP(\mathcal{T}_2)$ shall not rely on this particularity.

The symbol $||$ stands for "parallel". Points L_n and I_n on Figure 3 are constructed such that both $D(X, I_n)||D(A_n, C_n)$ and $D(X, L_n)||D(B_n, D_n)$. Shortening notations, if $z \in [0,1[$, we let $2^n z$ stand for $2^n z \mod 1$. Then undergraduate computations lead to

$$\begin{cases} \beta_n(X) := d(D_n, I_n) = \frac{1}{\sqrt{2}2^n}\left(1 - |2^n x - 2^n y|\right); \\ \gamma_n(X) := d(K_n, X) = \frac{1}{\sqrt{2}2^n}\left(1 - |2^n x - 2^n y| - |1 - (2^n x + 2^n y)|\right). \end{cases}$$

Putting $\ell_n := d(A_n, Z_n) = \frac{1}{\sqrt{2}2^n}$, we may define

$$t_n(x,y) = \left((1 - |2^n x - 2^n y|)^2, \left(\frac{1 - |2^n x - 2^n y| - |1 - (2^n x + 2^n y)|}{1 - |2^n x - 2^n y|} \right)^2 \right) \tag{4.1}$$

(which is nothing else but $\left(\left(\frac{\beta_n(X)}{\ell_n}\right)^2, \left(\frac{\gamma_n(X)}{\beta_n(X)}\right)^2\right)$).

Denote by $T(X,Y,Z)$ the region delimited by the triangle with edges X, Y, and Z in $I\!R^2$. Then we can easily observe (use Figure 3 and Eq. (4.1)) that for $d, d' \in]0,1]$,

$$t_n^{-1}((0,d) \times (0,d')) \cap T(A_n, Z_n, D_n) = T(J_n, X, K_n)$$

if $t_n(X) = (d, d')$. Then by construction $m_2(T(J_n, X, K_n)) = dd'm_2(T(A_n, Z_n, D_n))$. Thus we have

Lemma 4.2. *The sequence $(t_n)_{n\geq 0}$ defined in Eq. (4.1) is in $ABP(\mathcal{T}_2)$.*

Now we turn to the estimation of the $\alpha_{n,m}$'s (cf. Theorem 2.1.). Chosen $A = (0,d)\times(0,d') \subseteq C$ let us introduce $V_n = t_n^{-1}((0,d) \times (0,1))$ and $H_n = t_n^{-1}((0,1) \times (0,d'))$, $n \geq 0$, as in Example 4.1. From Definition 2.1 and Eq. (4.1) we deduce $\partial_{n+m}(E_n) \subseteq \partial_{n+m}(V_n) \cup \partial_{n+m}(H_n)$. Drawing a picture at first shows that $m_2(\partial_{n+1}(H_n) \cap P_{n,i}) \leq \frac{3}{4}$, where $P_{n,i} \in \mathcal{P}_n$. And the same holds for V_n. Again this repeats inductively and we get

$$\max\left(m_2(\partial_{n+m}(V_n)), m_2(\partial_{n+m}(H_n))\right) \leq \left(\frac{3}{4}\right)^m,$$

and henceforth $\alpha_{n,m} \leq 2\left(\frac{3}{4}\right)^m$. In view of Corollary 2.1. we obtain

Theorem 4.2. *For m_2 a.e. $(x, y) \in C$, the sequence $(t_n(x, y))$ defined in Eq.* (4.1) *is uniformly distributed in $(0, 1)^2$. Moreover, for such points (x, y), any $d, d' \in [0, 1]$, any $\varepsilon > 0$,*

$$\left|\frac{1}{N}\sum_{0\leq n<N} \mathbf{1}_{(0,d)\times(0,d')}(t_n(x, y)) - dd'\right| = \mathcal{O}\left(\frac{\log^{\frac{3}{2}+\varepsilon} N}{\sqrt{N}}\right).$$

5. Hints towards Cantor-Bernstein lemmas, central limit theorems, or a.s. cnvergence in L^p spaces

A typical example for "Cantor-Bernstein" lemmas is [Kh, Thm 32]. Now using [Khi], bottom of page 65, and "barycentrical positions" as in Theorem 3.1., we propose

Exercise 5.1. *Give a new proof with renormalization technics of* [Kh, Thm 32]. *State a "Cantor-Bernstein" type lemma for Cantor products* [La]. *And for dyadic expansions? Or Example 4.1.?*

Using technics from [Ko] and [Pe], concerning the central limit theorem, we propose

Exercise 5.2. *With the barycentrical positions of Theorem 3.1., for the partitions associated to Cantor products* [La], *find conditions on Fourier coefficients of $f \in L^2(\mathbb{T})$ (assume $\int_{\mathbb{T}} f dm = 0$) ensuring that $f(t_n(\cdot))$ obeys a central limit theorem.*

Exercise 5.3. *Using* [RoWi, Chap. VI], *find a sequence (m_n) of integers such that $m_n | m_{n+1}$ and there exists an $f \in L^\infty(\mathbb{T})$ for which, setting $t_n(x) = m_n x \mod 1$, $(\frac{1}{N}\sum_1^N f(t_n(x)))_N$ fails to converge m-a.s.. Also apply to this case* [KSZ] *in order to guarantee that for $f \in L^2(\mathbb{T})$ satisfying some Fourier conditions, the above m-a.s. convergence holds.*

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