Remarks on the Delange-Coquet formula

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Abstract. We give a dynamical model for generalized Delange-Coquet summation formulas, extending the concept. The acting semi-group is free with 2 generators, the action is strictly ergodic and the convergence is uniform.

I. Introduction

Let $\mathbb{N}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ be as usual. Let $q \geq 2$ and $A_q = \{0, 1, \ldots, q-1\}$. If $n \in \mathbb{N}$, the q-expansion of n is $n = \sum_{i=0}^{\infty} e_i(n) \cdot q^i$, where $e_i(n) \in A_q$. Let A_q^* be the set of words over alphabet A_q. When $w \in A_q^*$, $|w|$ denotes its length. For $w \neq 0^{|w|}$ and $n \in \mathbb{N}$, let

$$\rho_w(n) = \#\left\{j \geq 0,\ e_j(n)e_{j+1}(n)\cdots e_{j+|w|-1}(n) = w\right\}.$$

Finally if $N \in \mathbb{N}^*$, let $S_w(N) = \frac{1}{N}\sum_{n=0}^{N-1} \rho_w(n)$. This is the so-called *Delange-Coquet sum* associated to pattern w in q basis. Using analytic number theory (cf. [Kir], [Del] or [Coq]), one has $S_w(N) = \frac{\log N}{\log q}\frac{1}{q^{|w|}} + F_w(\frac{\log N}{\log q}) + \frac{D_w(N)}{N}$, where F_w is continuous 1-periodic, and D_w is bounded periodic.

In the following section, we give a dynamical form to $S_w(N)$ (cf. $(*)$). Lemma 1. is elementary, but useful in the proof of Theorem 1.. Lemma 2. describes the action for the strictly ergodic dynamical model. Finally Theorem 1. states the uniform convergence of the extended sums (cf. $(**)$).

II. The dynamical model

Let $G_q = \lim\limits_{\leftarrow} \mathbb{Z}_{q^t}$ be the group of q-adic integers ([He-Ro]). We identify G_q to $A_q^{\mathbb{N}}$. Then Haar measure h on G_q is the product measure of the uniform one on A_q. Let $\Phi : \mathbb{N} \to A_q^{\mathbb{N}}$ be such that $\Phi(n) = (e_i(n))_{i\geq 0}$. Map Φ is one to one, hence carries translation by 1 on $\mathbb{N}$ to a map on $\Phi(\mathbb{N})$. With the product topology, it is uniformly continuous, thus has a unique uniformly continuous extension on $\overline{\phi(\mathbb{N})} = G_q$; this is the

$^{(\triangle)}$ Research partially supported by the Bern Universität, Institut für Mathematik.

translation τ by generator $1^* = (1,0,0,\ldots)$. It is strictly ergodic and h-invariant. On G_q define the shift S (also h-invariant) by $S(e_i)_{i\geq 0} = (e_{i+1})_{i\geq 0}$ and let the characteristic function $1_w(\cdot)$ of w be defined by $1_w((e_i)_{i\geq 0}) = 1$ if $e_0 \cdots e_{|w|-1} = w$, 0 otherwise. Let $0^* = (0,0,\ldots)$. Then if $w \neq 0^{|w|}$,

$$(*) \qquad S_w(N) = \frac{1}{N} \sum_{i=0}^{[\frac{\log(N-1)}{\log q}]} \left(\sum_{n=0}^{N-1} 1_w\big(S^i(\tau^n(0^*))\big) \right).$$

Lemma 1. For any $g \in G_q$, $i \in I\!N$, and $w \in A_q^*$, the sequence $\big(1_w(S^i(\tau^n(g)))\big)_{n\geq 0}$ is *periodic with period* $q^{i+|w|}$. Furthermore, $\sum_{n=0}^{q^{i+|w|-1}} 1_w(S^i(\tau^n(g))) = q^i$.

Lemma 2. Let K_2 be the free semi-group with 2 generators. Then it acts *freely* on G_q, taking a generator to S and the other to τ. Moreover, the action is strictly ergodic.

Proofs. Lemma 1 is straightforward. For Lemma 2, we shall only sketch it : for instance, let $\mathcal{R}: \ S\circ\tau = \tau\circ S$. To show that $\mathcal{R}$ is false on G_q, we exhibit a subset $U(\mathcal{R})$ of positive Haar measure in G_q such that $\mathcal{R}$ is false on $U(\mathcal{R})$; take $U(\mathcal{R}) = \{g = (g_t)_{t\geq 0} \in G_q;\ g_0 = 0\}$. The strict ergodicity is evident on compact monothetic group G_q with translation by a generator. ■

Lemma 1 suggests to extend to arbitrary $g \in G_q$ the summation process $(*)$: given $N \in I\!N^*$, $g \in G_q$, and $0 \leq i \leq [\frac{\log(N-1)}{\log q}]$, let

$$(**) \qquad \xi_w(N,g,i) = \sum_{n=0}^{N-1} 1_w\big(S^i(\tau^n(g))\big) \ \text{ and } \ S_w(N,g) = \sum_{i=0}^{[\frac{\log(N-1)}{\log q}]} \xi_w(N,g,i).$$

Theorem 1. For any $w \in A_q^*$ and $g \in G_q$,

$$\lim_{N\to\infty} \frac{\log q}{N\log N} S_w(N,g) = \frac{1}{q^{|w|}} \qquad \left(= \int_{G_q} 1_w(g)dh(g) \right),$$

uniformly in $g \in G_q$. With $g = 0^*$, it follows that $\lim_{N\to\infty} \frac{\log q}{\log N} S_w(N) = \frac{1}{q^{|w|}}$.

Proof. If $0 \le i \le [\frac{\log(N-1)}{\log q}]$, let $N-1 = N_i q^{i+|w|} + R_i$, where $0 \le R_i < q^{i+|w|}$. Then (cf. (**)) $N_i q^i \le \xi_w(N,g,i) < \frac{N-1}{q^{|w|}} + q^i$. Summing over all $0 \le i \le [\frac{\log(N-1)}{\log q}]$, we deduce

$$\sum_{i=0}^{[\frac{\log(N-1)}{\log q}]} N_i q^i \le \sum_{i=0}^{[\frac{\log(N-1)}{\log q}]} \xi_w(N,g,i) < \left(\sum_{i=0}^{[\frac{\log(N-1)}{\log q}]} q^i \right) + \frac{1}{q^{|w|}}(N-1)\left([\frac{\log(N-1)}{\log q}] + 1 \right).$$

Since $N_i q^{i+|w|} > N - 1 - q^{i+|w|}$, $\displaystyle \frac{q^{|w|}}{N([\frac{\log(N-1)}{\log q}]+1)} \sum_{i=0}^{[\frac{\log(N-1)}{\log q}]} N_i q^i \ge 1 - \mathcal{O}(\frac{q^{|w|}}{\log N})$, and

$$\frac{q^{|w|}}{N([\frac{\log(N-1)}{\log q}]+1)} \left(\sum_{i=0}^{[\frac{\log(N-1)}{\log q}]} q^i + \frac{N-1}{q^{|w|}} \left([\frac{\log(N-1)}{\log q}] + 1 \right) \right) \le 1 + \mathcal{O}'(\frac{q^{|w|}}{\log N}),$$

where the constants in the $\mathcal{O}$ and $\mathcal{O}'$ do not depend on $g \in G_q$. ■

Remark. The free action of K_2 (Lemma 2.) involves a sequence of "rectangles" (cf. (*)) ; it satisfies Tempel'man's conditions except the covering one (cf. [Kre, p. 225]). From Lemma 1. one can see that the shape of these rectangles is necessary in some sense.

Question. Does a dynamical model hold for the summation process in the Fibonnacci expansion case (cf. [Gr-Li-Ti] and [Gr-Ti]) ?

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