A NON-REGULAR TOEPLITZ FLOW WITH PRESET PURE POINT SPECTRUM

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Abstract. Given an arbitrary countable subgroup σ_o of the torus, containing infinitely many rationals, we construct a strictly ergodic 0-1 Toeplitz flow with pure point spectrum equal to σ_o. For a large class of Toeplitz flows certain eigenvalues are induced by eigenvalues of the flow Y which can be seen along the aperiodic parts.

Introduction

In this paper we continue the study of Toeplitz flows initiated in 1984 by S. Williams in her work [W]. Toeplitz sequences have been known earlier (e.g. [O], [G-H], [J-K]), but it is the construction of Williams that is exploited in most of later works on Toeplitz sequences (e.g. [B-K1], [D], [B-K2], [I-L], [D-K-L], [I]). Spectral properties of Toeplitz flows have been studied in [I-L] and [I]. In this note we develop the method introduced by A. Iwanik in [I]. Each eigenvalue γ obtained there satisfies certain equation formulated in Section I of this paper as (3). In [I], however, this equation remains unsolved, and an irrational γ is obtained by constructing uncountably many Toeplitz flows with different eigenvalues.

We have succeeded in solving the equation (3) simultaneously for an arbitrary countable set of γ's. This enables us to prove the existence of strictly ergodic Toeplitz flows with an arbitrarily preset pure point spectrum containing infinitely many rationals.

Section I contains slightly modified formulations of the results of [I]. We rid the constructions of technical details used in [I] to produce uncountably many sequences. For a large class of Toeplitz flows we identify certain eigenvalues not arising from the maximal uniformly continuous factor. We also adapt the cohomology statement of [I] to the countable product of tori.

Section II is devoted to presenting how equation (3) can be solved for an arbitrary countable set of γ's.

In Section III we put the previous theorems together to obtain the desired Toeplitz flow with the preset spectrum. Attention is paid to strict ergodicity.

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Preliminaries

Let (X, μ, T) be a standard probability measure preserving dynamical system. The *point spectrum* $\sigma_o(T)$ is the set of all eigenvalues of the induced unitary operator U_T on $L^2(\mu)$. We say that the dynamical system has *pure point spectrum,* if $\sigma_o(T)$ supports the entire spectral measure of U_T. If in addition μ is ergodic, then $\sigma_o(T)$ is a countable subgroup of the torus and (X, μ, T) is measure-theoretically isomorphic to (G, λ, g), where G is the dual compact monothetic group to $\sigma_o(T)$ with Haar measure λ and rotation by a generator g.

A *Toeplitz sequence* is a non-periodic element $\eta \in \Sigma^{\mathbf{Z}}$ (Σ a finite set), such that

$$(\forall n \in \mathbf{Z})(\exists p \in \mathbf{N})(\forall m \in \mathbf{Z}) \;\; \eta(mp+n) = \eta(n)$$

(see [W] for a general reference on Toeplitz sequences). For each Toeplitz sequence there exists a sequence of periods $(p_j)_{j\in\mathbf{N}}$ with $p_j | p_{j+1}$ for each j, defining a partition of $\mathbf{Z}$ into p_j-periodic sets Z_j along which η is p_j-periodic but not periodic with any smaller period. Every such sequence (p_j) is called a *period structure* for η. Each subsequence of (p_j) is again a period structure for η. A Toeplitz sequence is called *regular* if the sum d of the densities in $\mathbf{Z}$ of Z_j equals 1.

It is known that the orbit closure $\overline{O}(\eta)$ of a Toeplitz sequence η is minimal for the shift transformation S. Now, if η is regular, then $(\overline{O}(\eta), S)$ is *strictly ergodic* i.e., in addition to minimality it carries a unique invariant (probability) measure. In this case, almost all elements of $\overline{O}(\eta)$ are also Toeplitz sequences. In the non-regular case each of the (possibly many) invariant measures is carried by the set $W(\eta)$ of (non-Toeplitz) elements with doubly infinite aperiodic part. We define

$$Y(\eta) = \overline{\{y(\omega) : \omega \in W(\eta)\}},$$

where $y(\omega)$ is the sequence to be read along the aperiodic part of ω (position zero of $y(\omega)$ is defined by the smallest positive position in the aperiodic part of ω). Of course, $Y(\eta) \subset \Sigma^{\mathbf{Z}}$ is shift-invariant. Since $W(\eta)$ is not closed, in the general case we have no guarantee that $\{y(\omega) : \omega \in W(\eta)\}$ is itself a closed set. However, in the further constructions of this paper it is the case, so the closure in the above definition can as well be omitted.

The maximal uniformly continuous factor of $(\overline{O}(\eta), S)$ appears to be topologically isomorphic with the unit-rotation on the group of p-adic integers, where p stands for (p_j), a period structure of η. If G_p is viewed as a compactification of $\mathbf{Z}$, then by C_j we denote the closure of Z_j. The sets C_j are disjoint and each of them is a union of some number l_j of cosets of the form $H_j + k$, where $H_j = p_j G_p$, $0 \le k < p_j$. We define

$$C = \dot{\bigcup_{j\ge 1}} C_j \; . \tag{1}$$

Note that

$$d = \sum_{j\ge 1} \frac{l_j}{p_j} = \mu_p(C),$$

where μ_p is the Haar measure on G_p. (Practically, we construct Toeplitz sequences by induction, filling in the step j some l_j yet unfilled positions of the interval $[0, p_j)$, and repeating the pattern with the period p_j).

There is a natural Borel measurable mapping ψ from $W(\eta)$ into the product $G_p \times Y(\eta)$ sending ω to (h, y), where h is the image of ω by the maximal uniformly continuous factor, and $y = y(\omega)$. The shift S then corresponds to a piecewise power skew product transformation

$$S_C(h, y) = (h + 1, S^{h \notin C} y),$$

where "$h \notin C$" denotes the logical 0-1-value, $S^1 = S$ and $S^0 = \mathrm{Id}$ (the identity map). If η is a Toeplitz sequence "constructed from a subshift Y", then $Y(\eta) = Y$, and for each invariant measure ν on Y, there exists an invariant measure λ on $(\overline{O}(\eta), S)$ such that ψ becomes a measure-theoretical isomorphism between $(\overline{O}(\eta), \lambda, S)$ and $(G_p \times Y, \mu_p \times \nu, S_C)$ (see [W, Theorem 4.5]). In the strictly ergodic case we need not refer to [W]. Sufficient is the following easy observation:

Lemma 1. *If the piecewise power skew product $(G_p \times Y(\eta), S_C)$ is strictly ergodic then $(\overline{O}(\eta), S)$ is strictly ergodic and ψ is a measure-theoretical isomorphism between these flows.*

Proof. The set $W(\eta)$ carrying all invariant measures of $(\overline{O}(\eta), S)$ is mapped by ψ in a 1-1 way onto an invariant subset of $G_p \times Y(\eta)$. By strict ergodicity, this must be a full measure subset for the unique invariant measure. Thus $W(\eta)$ carries a unique invariant measure, too. □

Inspired by [I], we will assume that for each $j \in \mathbf{N}$

$$\begin{aligned} &\text{either} \quad C_j \subset H_j \cup (H_j + 1) \cup \cdots \cup (H_j + r_j - 1) \\ &\text{or} \quad C_j \subset (H_j - r_j) \cup (H_j - r_j + 1) \cup \cdots \cup (H_j - 1) \end{aligned}$$

for a sequence (r_j) such that

$$\sum_{j \geq 1} \frac{r_j}{p_j} < \infty. \tag{2}$$

As not hardly seen, (2) already implies non-regularity of η. Nevertheless, we need the flow $(\overline{O}(\eta), S)$ to be strictly ergodic, in order to insure that the measure-theoretical notions do not depend on the choice of the invariant measure (except for Theorem 1, which holds separately for each invariant measure ν on $Y(\eta)$).

I. Iwanik's equation

We will view the torus $\mathbf{T}$ additively, as parameterized by the interval $[-\frac{1}{2}, \frac{1}{2})$. It has to be noted that in this setting the expression $\frac{nx}{m}$ for $x \in \mathbf{T}$ ($m > n$, some positive integers) does not have a definite meaning, so we shall use

$$\frac{(nx)_{\mathbf{T}}}{m}$$

to indicate that dividing by m is applied to the element of $\mathbf{T}$ arising from nx. Observe that then the above function of x is linear on each interval along which $(nx)_{\mathbf{T}} \in (-\frac{1}{2}, \frac{1}{2})$. In other expressions we can safely skip the indicator "$(\ )_{\mathbf{T}}$". The absolute value function is also applied to the element of $\mathbf{T}$ in the above parameterization, so that for $x \in \mathbf{T}$, $|nx|$ is the distance of the real number nx from the nearest integer. In our constructions complex numbers belonging to the following set (for a given Toeplitz sequence) will play the crucial role:

$$A = \{x \in \mathbf{T} \ : \sum_{j \geq 1} |l_j x| < \infty\},$$

where (l_j) is as described earlier for Toeplitz sequences. It is worth noticing (and not hard to see) that the set A has always Lebesgue measure zero. It can be countable (e.g. for (l_j) a geometric sequence), but if l_j increases suffitiently quickly then A contains a Cantor set.

Note that now a $\mathbf{T}$-valued function f is an eigenfunction pertaining to an eigenvalue $\alpha \in \mathbf{T}$ if $f(S\omega) = f(\omega) + \alpha$.

Theorem 1. *Let $(\overline{O}(\eta), \lambda, S)$ be a Toeplitz flow measure-theoretically isomorphic via the map ψ to the skew product $(G_p \times Y(\eta), \mu_p \times \nu, S_C)$ for some invariant measure ν on $Y(\eta)$. Assume also (2). Let $\alpha \in A$ be an eigenvalue for $(Y(\eta), \nu, S)$ corresponding to an eigenfunction $f : Y(\eta) \to \mathbf{T}$. Then*

$$\gamma(\alpha) = \alpha - \sum_{j\geq 1} \frac{(l_j\alpha)_{\mathbf{T}}}{p_j} \tag{3}$$

is an eigenvalue for $(\overline{O}(\eta), \lambda, S)$. The corresponding eigenfunction has the form $\tilde{f}(\omega) = g(h) + f(y)$, where $(h, y) = \psi(\omega)$ and $g : G_p \to \mathbf{T}$.

Proof. We need to construct the function $g : G_p \to \mathbf{T}$. Fix $j \in \mathbf{N}$. Recall that C_j is a union of l_j cosets $H_j + k$. Now, for $h \in H_j$ we define

$$g_j(h) = 0 \tag{4}$$

and

$$g_j(h+k+1) = \begin{cases} g_j(h+k) - \frac{(l_j\alpha)_{\mathbf{T}}}{p_j} & \text{if } \ h+k \notin C_j \\ g_j(h+k) - \frac{(l_j\alpha)_{\mathbf{T}}}{p_j} + \alpha & \text{if } \ h+k \in C_j \end{cases} \tag{5}$$

for $k = 0, 1, \ldots, p_j - 1$. Observe that the formula (5) applied to $k = (p_j - 1)$ yields

$$g_j(h+p_j) = g_j(h) - p_j\frac{(l_j\alpha)_{\mathbf{T}}}{p_j} + l_j\alpha = 0,$$

which agrees with (4), since $H_j + p_j = H_j$. Also note that $h + k \notin C_j$ for $p_j - r_j$ consecutive values of k either ending with $p_j - 1$ or starting with 0, and hence g_j differs from zero at the corresponding cosets by at most

$$|(p_j - r_j)\frac{(l_j\alpha)_{\mathbf{T}}}{p_j}| \leq |l_j\alpha| \ ,$$

which is summable over j since $\alpha \in A$. The remaining part of G_p has measure $\frac{r_j}{p_j}$, also assumed in (2) to be a summable sequence. It is now obvious, that the function

$$g(h) = \sum_{j\geq 1} g_j(h)$$

is well defined as a convergent series on a set of measure 1 in G_p. On the same set, by (5),

$$g(h+1) = \begin{cases} g(h) - \sum_{j\geq 1} \frac{(l_j\alpha)_{\mathbf{T}}}{p_j} & \text{if } \ h \notin C \\ g(h) - \sum_{j\geq 1} \frac{(l_j\alpha)_{\mathbf{T}}}{p_j} + \alpha & \text{if } \ h \in C, \end{cases} \tag{6}$$

because in the second case $h \in C_j$ for exactly one j.

Finally, as announced, we define $\tilde{f}(\omega) = g(h) + f(y)$ and we check

$$\begin{aligned} \tilde{f}(S\omega) &= g(h+1) + f(S^{h\notin C}y) \\ &= \begin{cases} g(h) - \sum_{j\geq 1} \frac{(l_j\alpha)_{\mathbf{T}}}{p_j} + f(Sy) & \text{if } \ h \notin C \\ g(h) - \sum_{j\geq 1} \frac{(l_j\alpha)_{\mathbf{T}}}{p_j} + \alpha + f(y) & \text{if } \ h \in C \end{cases} \\ &= \tilde{f}(\omega) + \alpha - \sum_{j\geq 1} \frac{(l_j\alpha)_{\mathbf{T}}}{p_j} \ , \end{aligned}$$

almost everywhere for the product measure, as desired. □

Note that if α is rational then $\alpha \in A$ implies $(l_j\alpha)_{\mathbf{T}} = 0$ starting from some j_o. Thus $\gamma(\alpha)$ is rational, too.

The equation (3) along with the function g of Theorem 1 can be also used to produce measure-theoretical isomorphisms by cohomology between group extensions over G_p by $\mathbf{T}^\infty$, where $\mathbf{T}^\infty$ denotes the product of countably many tori. This will be used later for further investigations of Toeplitz flows.

We fix the sequence (p_j) (and by that the group G_p) and an open set $C \subset G_p$ represented by the formula (1) and satisfying (2). The sequence (l_j) and the set A are therefore also determined. We do not need to further specify the Toeplitz sequence η.

Theorem 2. *Let $\boldsymbol{\alpha} = (\alpha_i)_{i\in\mathbf{N}}$ with $\alpha_i \in A$ for each i. Let $\gamma_i = \gamma(\alpha_i)$, and let $\boldsymbol{\gamma} = (\gamma_i)_{i\in\mathbf{N}}$. If $1, \gamma_1, \gamma_2, \ldots$ are rationally independent then the group extension $(G_p \times \mathbf{T}^\infty, T_C)$, where*

$$T_C(h, \mathbf{x}) = (h + 1, \mathbf{x} + \boldsymbol{\alpha}^{h\notin C}),$$

is strictly ergodic and measure-theoretically isomorphic to the rotation of $G_p \times \mathbf{T}^\infty$ by the generator $1 \times \boldsymbol{\gamma}$.

Proof. As in [I], we show that the cocycle $\boldsymbol{\alpha}^{h\notin C}$ is cohomologous to the constant cocycle $\boldsymbol{\gamma}$. To this end we check that the cohomology function $\mathbf{g} : G_p \to \mathbf{T}^\infty$ is given by

$$\mathbf{g}(h) = (g_i(h))_{i\in\mathbf{N}} \ ,$$

where g_i now denotes the function g constructed as in the proof of Theorem 1 for α_i and γ_i. In fact, by the property (6) of each g_i, we have

$$(\boldsymbol{\alpha}^{h\notin C} + \mathbf{g}(h+1) - \mathbf{g}(h))_i$$

$$= \begin{cases} \alpha_i - \sum_{j\ge 1} \frac{(l_j\alpha_i)_{\mathbf{T}}}{p_j} & \text{if} \quad h \notin C \\ -\sum_{j\ge 1} \frac{(l_j\alpha_i)_{\mathbf{T}}}{p_j} + \alpha_i & \text{if} \quad h \in C \end{cases}$$

$$= \gamma_i \ ,$$

μ_p-almost everywhere, which is the cohomology equation.

Since the rotation on $G_p \times \mathbf{T}^\infty$ is ergodic, so is the group extension T_C, hence, by a theorem of Furstenberg [F], the later is also strictly ergodic. □

The group extension $(G_p \times \mathbf{T}^\infty, T_C)$ carries at least as many invariant measures as $(\mathbf{T}^\infty, \boldsymbol{\alpha})$ (corresponding product measures are T_C-invariant), hence we immediately deduce that

Corollary 1. *With the assumptions of Theorem 2, the numbers $1, \alpha_1, \alpha_2, \ldots$ are rationally independent.* □

(The above fact can be also deduced for α_i's belonging to A directly from (3). The assumption (2) is not essential.)

Remark 1. With a slight modification of the set C, Theorem 2 also holds if we add one (finitely many reduces to one) rational number $\alpha_0 = \frac{1}{n}$, where n is relatively prime with all the p_j's. The group $\mathbf{T}^\infty$ is then replaced by $\mathbf{Z}_n \times \mathbf{T}^\infty$ ($\mathbf{Z}_n$ is viewed as a subgroup of $\mathbf{T}$). Next, all l_j's must be chosen multiples of n, hence we obtain $\gamma_0 = \alpha_0$. The cohomology function on the added axis is then $g_0 = 0$. The ergodicity and strict ergodicity are maintained.

II. Solving Iwanik's equation

Let the group G_p be given. By p-rationals we will mean the elements of the spectrum of the group rotation $(G_p, 1)$. For technical reasons we need to introduce a notation for the number of "unfilled positions". And so, for given sequences (p_j) and (l_j) we define

$$m_j = p_j(1 - \sum_{k \le j} \frac{l_k}{p_k}).$$

Let (δ_j) be a summable sequence with $0 < \delta_j < \frac{1}{2}$ for each j.

Theorem 3. *For every sequence (γ_i) of elements of $\mathbf{T}$ there exist*
(*a*) *increasing sequences (l_j) and (p_j) with (p_j) defining G_p, and such that for each $j \in \mathbf{N}$*

$$l_j \text{ is a multiple of } m_{j-1}, \quad \text{and} \quad l_j < \frac{\delta_j p_j}{p_{j-1}}, \tag{7 for j}$$

(*b*) *a sequence (β_i) of p-rationals, and*
(*c*) *a sequence (α_i) such that $\alpha_i \in A$ and $\gamma(\alpha_i) = \gamma_i + \beta_i$ for each $i \in \mathbf{N}$.*

Proof. Fix a sequence (ϵ_j) of positive numbers, such that

$$\epsilon_1 < \frac{1}{6} \quad \text{and} \quad \sum_{j>k} \epsilon_j < \epsilon_k \ \ (k \ge 1). \tag{8}$$

We shall inductively find the numbers l_j and p_j satisfying (7 for j), define β_j, and construct a triangular array $(\alpha_{i,j})_{j \in \mathbf{N}, 1 \le i \le j+1}$ such that for every pair i, j with $i \le j$ hold

$$\alpha_{i,j} - \sum_{k \le j} \frac{(l_k \alpha_{i,j})_{\mathbf{T}}}{p_k} = \gamma_i + \beta_i \ , \tag{9 for j}$$

and

$$|l_k \alpha_{i,j}| < 2\epsilon_k + \epsilon_{k+1} + \cdots + \epsilon_j \qquad \text{for all } 1 \le k \le j. \tag{10 for j}$$

Step 1
Let $\beta_1 = 0$ and find $l_1 \ge 1$ such that

$$|l_1 \gamma_1| < \epsilon_1$$

(this is possible for both rational and irrational γ_1). Then let p_1 be an element of the sequence defining G_p, such that $l_1 < \delta_1 p_1$, as in (7 for 1) (assign $p_0 = m_0 = 1$). Observe that for $x \in [\gamma_1 - \frac{\epsilon_1}{l_1}, \gamma_1 + \frac{\epsilon_1}{l_1}]$ we have $(l_1 x)_{\mathbf{T}} \in (-2\epsilon_1, 2\epsilon_1) \subset (-\frac{1}{2}, \frac{1}{2})$, so that the function

$$x - \frac{(l_1 x)_{\mathbf{T}}}{p_1}$$

increases linearly (with slope $1 - \frac{l_1}{p_1}$) from a value $< \gamma_1$ to a value $> \gamma_1$ (seen by an elementary calculation). Thus, there exists an $\alpha_{1,1} \in [\gamma_1 - \frac{\epsilon_1}{l_1}, \gamma_1 + \frac{\epsilon_1}{l_1}]$ with

$$\alpha_{1,1} - \frac{(l_1 \alpha_{1,1})_{\mathbf{T}}}{p_1} = \gamma_1. \tag{9 for 1}$$

Of course, we have

$$|l_1\alpha_{1,1}| < 2\epsilon_1. \tag{10 for 1}$$

Step j+1
Throughout description of this step $j \geq 1$ has a fixed value, while i and k range between 1 and j (later we also admit $i = j+1$ and $k = j+1$). Suppose we have already defined all the numbers $\beta_i,\ l_k,\ p_k$ and $\alpha_{i,j}$, so that the requirements on l_k and p_k of (7 for k) as well as the conditions (9 for j) and (10 for j) are satisfied. First notice that

$$|l_k x| < 2\epsilon_k + \epsilon_{k+1} + \cdots + \epsilon_j \tag{11}$$

holds for each k if x is in some open interval U_j around 0. In particular, by (8), $(l_k x)_{\mathbf{T}} \in (-\frac{1}{2}, \frac{1}{2})$ thus the function

$$x - \sum_{k \leq j} \frac{(l_k x)_{\mathbf{T}}}{p_k}$$

increases linearly (with positive slope $1 - \sum_{k\leq j} \frac{l_k}{p_k}$) on U_j. So, it is possible to find $x_o \in U_j$ and a p-rational β_{j+1}, with

$$x_o - \sum_{k \leq j} \frac{(l_k x_o)_{\mathbf{T}}}{p_k} = \gamma_{j+1} + \beta_{j+1}. \tag{12}$$

We let $\alpha_{j+1,j} = x_o$, and so, by (11) and (12), the conditions (9 for j) and (10 for j) are additionally satisfied with $i = j+1$.

Now, we repeat the procedure of step 1. Find $l_{j+1} > l_j$, a multiple of m_j, such that

$$|l_{j+1}\alpha_{i,j}| < \epsilon_{j+1} \qquad \text{for all } \ 1 \leq i \leq j+1 \tag{13}$$

(this is possible regardless of the rational independence of the $\alpha_{i,j}$'s), and then pick p_{j+1} from the sequence defining G_p such that $l_{j+1} < \frac{\delta_{j+1}p_{j+1}}{p_j}$, as required in (7 for j+1).

In the sequel i denotes a fixed integer between 1 and $j+1$.
For

$$x \in [\alpha_{i,j} - \frac{\epsilon_{j+1}}{l_{j+1}}, \alpha_{i,j} + \frac{\epsilon_{j+1}}{l_{j+1}}]$$

we have, by (10 for j),

$$|l_k x| < 2\epsilon_k + \epsilon_{k+1} + \cdots + \epsilon_j + \epsilon_{j+1} \qquad \text{for all } 1 \leq k \leq j. \tag{14}$$

Next, by (13), we also have

$$|l_{j+1}x| < 2\epsilon_{j+1}$$

on the same interval, which extends (14) to $k = j+1$. As before, by (8), the function

$$x - \sum_{k \leq j+1} \frac{(l_k x)_{\mathbf{T}}}{p_k}$$

increases linearly (with positive slope $1 - \sum_{k\leq j+1} \frac{l_k}{p_k}$) on the above interval, assuming at the endpoints values on opposite sides of $\gamma_i + \beta_i$ (easily calculated using (9 for j)) . Hence, there exists in this interval an $\alpha_{i,j+1}$ satisfying

$$\alpha_{i,j+1} - \sum_{k \leq j+1} \frac{(l_k \alpha_{i,j+1})_{\mathbf{T}}}{p_k} = \gamma_i + \beta_i. \tag{9 for j+1}$$

Of course, by (14), we also have

$$|l_k\alpha_{i,j+1}| < 2\epsilon_k + \epsilon_{k+1} + \cdots + \epsilon_{j+1} \quad \text{for all } 1 \leq k \leq j+1. \tag{10 for j+1}$$

This completes the induction.

It follows from the construction, that for fixed i the sequence $(\alpha_{i,j})_{j\geq i}$ is Cauchy, so

$$\alpha_i = \lim_j \alpha_{i,j}$$

exists. We easily notice that (10 for j) and (8) yield

$$|l_k \alpha_i| \leq \epsilon_k + \sum_{j\geq k} \epsilon_j < 3\epsilon_k \ ,$$

and so $\alpha_i \in A$. Further, by a standard argument with separately estimating a tail of a series and the corresponding finite sum, (9 for j) and (10 for j) imply that

$$\gamma(\alpha_i) = \gamma_i + \beta_i,$$

as desired. □

III. Strictly ergodic Toeplitz flow

At first, we need to represent an ergodic rotation of the infinite-dimensional torus by a subshift. Then we state our main result.

Lemma 2. *Let $(\mathbf{T}^\infty, \boldsymbol{\alpha})$ be ergodic. Then there exists a minimal 0-1-subshift (Y, S) and a continuous factor map $\phi : Y \to \mathbf{T}^\infty$ invertible except on a subset $F \subset \mathbf{T}^\infty$ of Haar measure zero.*

Proof. Let $0 = a_0 < a_1 < a_2 < \cdots < \frac{1}{2}$ be a sequence in $\mathbf{T}$. Next define

$$J = \bigcup_{n\geq 1} [a_{n-1}, a_n] \times [a_{n-2}, a_{n-1}] \times \cdots \times [a_0, a_1] \times \mathbf{T} \times \mathbf{T} \times \ldots .$$

It is easy to check that J and its complement constitute a topological generator for the flow $(\mathbf{T}^\infty, \boldsymbol{\alpha})$. The existence of a 0-1-subshift (Y, S) and a continuous factor map $\phi : Y \to \mathbf{T}^\infty$ invertible except on the subset

$$F = \bigcup_{n\in\mathbf{Z}} (\partial J + n\boldsymbol{\alpha})$$

follow from [D-G-S, sec. 15] (where ∂J denotes the boundary of J). Of course $\partial J \subset [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times \ldots$ is of Haar measure zero. By minimality of $\mathbf{T}^\infty$, Y can be chosen minimal. □

Theorem 4. *Let σ_o be a countable subgroup of $\mathbf{T}$ containing infinitely many rationals. Then there exists a strictly ergodic Toeplitz flow $(\overline{O}(\eta), S)$ with pure point spectrum σ_o.*

Proof. Every such subgroup σ_o is generated by the union of two sets: $\{\frac{1}{p_j} : j \in \mathbf{N}\}$ and $\{\gamma_i : i \in I\}$, where $p_j | p_{j+1}$ for each j, I is either empty or finite or countable, and (if I nonempty) the γ_i's are rationally independent. We proceed with the proof for $I = \mathbf{N}$, for the other cases see Remark 2 below.

Let G_p be the group of p-adic integers defined by the sequence (p_j). We now apply Theorem 3 (with some fixed sequence (δ_j)), to obtain a subsequence of (p_j) (from now on (p_j) will denote this subsequence), and sequences (l_j), (β_i), (α_i) with all the properties stated there. It is important that the set $\{\frac{1}{p_j} : j \in \mathbf{N}\} \cup \{\gamma_i + \beta_i : i \in I\}$ still generates the same group σ_o. From now on γ_i will denote $\gamma_i + \beta_i$. These γ_i's are rationally independent, too.

Let (Y, S) be a subshift (we will specify it a little later, at this moment we are more interested in defining the set C). We apply a simplified (compared to [W]) inductive construction of the Toeplitz sequence η from Y: in step j we fill the initial (for odd j) or terminal (for even j) l_j yet unfilled positions in $[0, p_j)$ using a block of length l_j appearing in Y, and repeat this pattern with period p_j. Of course, this induction also defines the set $C \subset G_p$. Observe that the last inequality of (7 for j) implies (2) (in our case $r_j \le l_j p_{j-1}$).

So, we can use Theorem 2, by which $(G_p \times \mathbf{T}^\infty, 1 \times \boldsymbol{\gamma})$, the monothetic group rotation with pure point spectrum σ_o, is measure-theoretically isomorphic to the strictly ergodic (with the product measure) group extension $(G_p \times \mathbf{T}^\infty, T_C)$, where T_C is defined by the cocycle $\boldsymbol{\alpha}^{h \notin C}$, and $\boldsymbol{\alpha}$ is the sequence of α_i's. By Corollary 1, the element $\boldsymbol{\alpha}$ is a generator of $\mathbf{T}^\infty$.

We can now specify the subshift (Y, S) to be the representation of $(\mathbf{T}^\infty, \boldsymbol{\alpha})$ of Lemma 2. The group extension $(G_p \times \mathbf{T}^\infty, T_C)$ is a topological factor, via $\mathrm{Id} \times \phi$, of the piecewise power skew product $(G_p \times Y, S_C)$, and this factor map is invertible except on the set $G_p \times F$ of product measure zero. Now, any two invariant measures on the skew product are mapped to the product measure on the group extension (strict ergodicity), hence they may differ only on the preimage of $G_p \times F$. This set however is of any such measure zero, so the difference is inessential. Thus strict ergodicity of $(G_p \times Y, S_C)$ is proved. Clearly, the map $\mathrm{Id} \times \phi$ provides also a measure-theoretical isomorphism.

Finally, by the fact that each l_j is a multiple of m_{j-1}, we have $Y(\eta) \subset Y$ (see proof of the analogous inclusion in [W, Lemma 4.3]). By minimality of Y we have equality, and Lemma 1 says that $(\overline{O}(\eta), S)$ is strictly ergodic and measure-theoretically isomorphic to $(G_p \times Y, S_C)$. □

Remark 2. For I finite the same proof applies (Theorem 2 works as well for finite products of tori, in Theorem 3 put $\gamma_i = 0$ for $i > \#I$). Any regular Toeplitz sequence over G_p works for $I = \emptyset$.

Remark 3. By essentially the same proof, if $n = p_1$ is relatively prime with $q_j = \frac{p_j}{p_1}$ for each $j \ge 2$, then we can obtain σ_o for a Toeplitz sequence over G_q. We let $\gamma_0 = \frac{1}{n}$ and replace $\mathbf{T}^\infty$ by $\mathbf{Z}_n \times \mathbf{T}^\infty$. Theorem 2 applies by Remark 1 with $\alpha_0 = \gamma_0$ (cf. [I, Theorem 2], which is identical with applying this Remark to the case $I = \emptyset$ of Remark 2).

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