SPECTRAL ISOMORPHISM OF MORSE FLOWS

T. Downarowicz, J. Kwiatkowski, Y. Lacroix

October 28, 1998

Introduction

Morse flows form a class of group extensions over adding machines. In general, if $(X \times G, T_\phi, \nu \times \lambda)$ is an ergodic group extension of an ergodic transformation (X, T, ν) (G compact abelian group with Haar measure λ and $\phi : X \to G$ a cocycle) then the space $L^2(\nu \times \lambda)$ decomposes into a product of T_ϕ-invariant subspaces L^2_γ where γ ranges over the group $\hat{G}$ dual to G, and $L^2_\gamma = \{f \otimes \gamma : f \in L^2(\nu)\}$.

If (X, T, ν) is a canonical factor of all its group extensions (e.g., if T has discrete spectrum or if T is prime), and two such group extensions are metrically isomorphic, then there exists a unique group automorphism $\hat{v}$ of $\hat{G}$, such that the induced spectral isomorphism sends each L^2_γ to $L^2_{\hat{v}(\gamma)}$ ([N], [J-L-M]).

The following question is of our interest: suppose that two ergodic group extensions of the same ergodic transformation are spectrally isomorphic, in this way that the isomorphism sends each L^2_γ to some $L^2_{\pi(\gamma)}$, where π is a permutation of $\hat{G}$. We will say that π *governs the spectral isomorphism.* Under what assumptions can π be chosen a group automorphism?

The group extensions provided by Morse cocycles have two convenient properties: the spectrum over each L^2_γ is simple, and the corresponding spectral measure is a generalized Riesz product. Such measures (if they are continuous) are either equivalent or orthogonal. This implies, that every spectral isomorphism between Morse flows is governed by a permutation π. If both cocycles have simple spectrum, then π is unique, and we can ask whether or not it is a group automorphism. In the opposite case we can ask about the existence of an automorphism among all permutations governing spectral isomorphisms between the flows.

In this paper we characterize spectral isomorphism between Morse flows in terms of combinatorial properties of the defining blocks (Theorem 4), and next we describe two classes of Morse flows, where the answer to the automorphism problem is always positive:
(1) $G = \mathbb{Z}_p$ is the cyclic group of prime order,
(2) G is arbitrary finite and the cocycle has an additional property AS
(in both cases we assume that the cocycle has "bounded lengths").

Due to the nature of the methods applied, we will concentrate mainly on the the symbolic representation of Morse flows, rather than the cocycle setup.

1991 *Mathematics Subject Classification.* 11K31, 28D05, 42A55, 47A35, 94A99.

Key words and phrases. Morse sequence, spectral isomorphism.

The first draft of this paper was written while the first two authors were visiting the Université de Bretagne Occidentale, April-May 1998,

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Basics

Let us now introduce the basic definitions and facts: as usually, $\mathbb{Z}$ denotes the set of all integers, while $\mathbb{N}$ stands for $\{1,2,3,\dots\}$. $\mathbb{T}$ denotes the unit circle on the complex plane, and $\bar{z}$ is the conjugate of z.

Let G be a finite abelian group denoted multiplicatively, with unity $\mathbf{1}$. As usual, by $\hat{G}$ we will denote the dual group of G, its elements (characters on G) will be denoted by the letters γ, γ', etc., while γ_0 will denote the trivial character (i.e., the unity of $\hat{G}$).

By a *block B of length $n \in \mathbb{N}$ over G* we shall mean a finite sequence $B = (b_0, b_1, ..., b_{n-1}) \in G^n$. A *sequence over G* has the form $B = (b_0, b_1, ...) \in G^{\mathbb{N}}$. We say that the block B is *symmetric* if $b_i = b_{n-1-i}$ for each $0 \le i < n$.

Definition 1. Let B be a block of length n over G. For $0 \le k \le n-1$ and $g \in G$ we denote

$$\mathrm{fr}_B(k,g) = \frac{1}{n}\#\{i : 0 \le i \le n-k-1, (b_i)^{-1} b_{i+k} = g\}.$$

Clearly, $\mathrm{fr}_B(0,g) = 1$ if $g = \mathbf{1}$ and 0 for other elements g.

Definition 2. Let $B = (b_0, b_1, ..., b_{n-1}) \in \mathbb{T}^n$, (such B will be called a *word*). The *aperiodic autocorrelation function* of B is defined on $\{0, 1, ..., n-1\}$ by

$$\Phi_B(k) = \frac{1}{n}\sum_{i=0}^{n-k-1} \overline{b_i} b_{i+k} = \sum_{z\in\mathbb{T}} z\ \mathrm{fr}_B(k,z).$$

Clearly, $\Phi_B(0) = 1$.

If A denotes a sequence over G (or $\mathbb{T}$) then we define $\mathrm{fr}_A(k,g)$ (and $\Phi_A(k)$) as the limit of $\mathrm{fr}_{A_n}(k,g)$ ($\Phi_{A_n}(k)$), where A_n is the initial block in A of length n. Of course, there is no guarantee that such limits exist.

Let $B = (b_0, b_1, ..., b_{n-1})$ and $A = (a_0, a_1, ..., a_{m-1})$ be two blocks over G. We define their *product* as the block $B \times A = (c_0, c_1, ..., c_{mn-1})$, by

$$c_{s+nt} = b_s a_t, \ \ 0 \le s < n, \ 0 \le t < m.$$

The above definition can be applied also to the case where A represents a sequence over G.

The following lemma appears in many variants in the literature (see e.g. [D-L]) and it says that the autocorrelation function of a product of words depends only on the autocorrelations of the component words.

Lemma 1. *Let $B = (b_0, b_1, ..., b_{n-1})$ and $A = (a_0, a_1, ..., a_{m-1})$ be two words (i.e. blocks over $\mathbb{T}$). For each $0 \le s < n$ and $0 \le t < m$ we have*

$$\Phi_{B\times A}(s+nt) = \Phi_B(s)\Phi_A(t) + \overline{\Phi_B(n-s)}\Phi_A(t+1),$$

(with the convention that $\Phi_B(n) = 0$). $\square$

Remark 1. The same holds true if A represents a sequence over $\mathbb{T}$ for which the autocorrelation function exists.

Definition 3. Let $(B_1, B_2, ...)$ be a sequence of blocks over a finite group G such that, for each $q \in \mathbb{N}$,
1) the length n_q of B_q is at least 2, and
2) $B_q(0) = \mathbf{1}$.
The (one-sided, generalized) *Morse sequence* A determined by the sequence of blocks (B_q) is defined as the coordinatewise limit of the words $A_q = B_1 \times B_2 \times \cdots \times B_q$ (convergence is granted by the condition $B_q(0) = \mathbf{1}$).

Let A be a Morse sequence over G. We define $X_A \in G^{\mathbb{Z}}$ as the set of all bi-infinite sequences x such that every block appearing in x appears infinitely many times in A. It is clear that X_A is closed and σ-invariant where σ is the left shift transformation

$$(\sigma x)_n = x_{n+1},$$

($x \in G^{\mathbb{Z}}$). The subshift (X_A, σ) will be called *the Morse flow generated by* A (or by the sequence of blocks $(B_1, B_2, ...)$). Morse flows have been extensively studied for their dynamical and spectral properties. We refer the reader to [G],[J],[K1],[K2],[K-S],[M]. For us it is important to know the following five facts:

Fact 1. (see [I-L] or [M] for similar statements) *A sufficient condition for a Morse flow generated by a sequence of blocks* $(B_1, B_2, \dots)$ *to be strictly ergodic is that there exists* $q_0 \geq 1$ *and* $\epsilon_A > 0$ *such that for each* $q \geq q_0$

$$fr_{A_q}(1, g) > \epsilon_A,$$

for all $g \in G$*, where* A_q *denotes the Morse sequence defined by the "truncated" sequence of blocks* $(B_{q+1}, B_{q+2}, \dots)$. $\square$

From now on we assume that our Morse sequence A satisfies the condition of Fact 1. This implies in particular that for every nontrivial character $\gamma \in \hat{G}$

$$\limsup_q |\Phi_{\gamma(A_q)}(1)| < 1 - \xi_A.$$

for some $\xi_A < 0$.

Remark 2. The above is a natural requirement and in most papers on Morse flows this or similar assumptions are made to ensure ergodicity of the flow and continuity of the interesting part of the spectrum. Our condition also ensures that the flow does not reduce to a Morse flow over a subgroup of G.

For spectral description of the Morse flow we need to consider the Hilbert space $L^2 = L^2(\nu_A)$, where ν_A is the unique invariant measure on X_A.

Fact 2. *We have the following decomposition*

$$L^2 = \bigoplus_{\gamma \in \hat{G}} L^2_\gamma,$$

where L^2_γ *is the* σ*-invariant subspace of* L^2 *defined by*

$$L^2_\gamma = \{f \in L^2 : f(gx) = \gamma(g) f(x) \text{ for each } g \in G \text{ and } x \in X_A\}$$

(gx *is obtained by multiplying all entries of* x *by* g*). Moreover, the spectrum of* σ *on each* L^2_γ *is:*

- simple,
- discrete, if $\gamma = \gamma_0$,
- continuous, for nontrivial characters.
The spectral type of σ on L^2_γ with $\gamma \neq \gamma_0$ is the same as that of the spectral measure $\mu_{(A,f_\gamma)}$ ($\mu_{(A,\gamma)}$ for short) of the function f_γ defined by

$$f_\gamma(x) = \gamma(x_0).$$

□

Recall, that the Fourier coefficients of the spectral measure $\mu_{(A,f)}$ of a function $f \in L^2$ are

$$\hat{\mu}_{(A,f)}(k) = \int z^k d\mu_{(A,f)} = \int \overline{f}(f \circ \sigma^k) d\nu_A.$$

If $f = f_\gamma$ (γ nontrivial) then, by continuity of f_γ and unique ergodicity, we can evaluate the integrals by taking averages along the trajectory of the element A. In this manner we obtain the following equalities:

Fact 3. *The Fourier coefficients of the spectral measures $\mu_{(A,\gamma)}$ coincide with the autocorrelations of the sequence obtained from A by applying the character γ:*

$$\hat{\mu}_{(A,\gamma)}(k) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \overline{\gamma(A(i))}\gamma(A(i+k)) = \Phi_{\gamma(A)}(k),$$

for each $k \in \mathbb{N}$ (and the autocorrelation functions exist for each $\gamma \in \hat{G}$). □

Fact 4. (see [K2] and [C-N] for similar statements) *Let A be a Morse sequence defined by a sequence of blocks $(B_1, B_2, \dots)$, satisfying the assumption of Fact 1. Then, for each nontrivial character $\gamma \in \hat{G}$ and each $q \in \mathbb{N}$, the measures $\mu_{(A,\gamma)}$ and $\mu_{(A_q,\gamma)}$ are equivalent (recall that A_q denotes the "truncated" Morse sequence defined by $(B_{q+1}, B_{q+2}, \dots)$).* □

In our study of spectral isomorphisms we will compare the behavior of the Morse flows defined by two Morse sequences, which we denote by A and A'. In our notation we will use the convention that all letters with a $'$ refer to the objects related to the Morse flow defined by A', corresponding to the objects denoted for A by the same letters without a $'$.

Fact 5. [K] *Let A and A' be two Morse sequences over G defined by two sequences of blocks (B_q) and (B'_q), respectively, with the same structure of lengths ($n_q = n'_q$ for each q). Let $\gamma, \gamma' \in \hat{G}$. Then the spectral measures $\mu_{(A,\gamma)}$ and $\mu_{(A',\gamma')}$ are either equivalent or orthogonal. If they are equivalent then*

$$||\mu_{(A_q,\gamma)} - \mu_{(A'_q,\gamma')}|| \to 0,$$

where $||\cdot||$ denotes the variation norm of measures □

As an immediate consequence of Fact 2 and the first statement of Fact 5 we obtain the following

Theorem 1. *The Morse flows defined by A and A' are spectrally isomorphic if and only if there exists a permutation π of $\hat{G}$ such that $\pi(\gamma_0) = \gamma_0$, and, for each $\gamma \in \hat{G}$, $\mu_{(A,\gamma)}$ and $\mu_{(A',\pi(\gamma))}$ are equivalent.* □

Spectral isomorphism and autocorrelations

In this section we will show that spectral equivalence between $\mu_{(A,\gamma)}$ and $\mu_{(A',\gamma')}$ mentioned in Theorem 1 can be tested by checking the autocorrelations of the defining blocks B_q and B'_q.

Theorem 2. *Suppose, with the assumptions of Fact 1, that $\mu_{(A,\gamma)}$ and $\mu_{(A',\gamma')}$ are equivalent for some γ and γ'. Then*

$$\max_{0\le s<n_q} |\Phi_{\gamma(B_q)}(s) - \Phi_{\gamma'(B'_q)}(s)| \to 0,$$

as q tends to infinity.

Proof. The statement holds trivially for $\gamma = \gamma' = \gamma_0$. Assume $\gamma \neq \gamma_0$ (then also $\gamma' \neq \gamma_0$). Denote $\epsilon_q = \max_{k\in\mathbb{N}} |\Phi_{\gamma(A_q)}(k) - \Phi_{\gamma'(A'_q)}(k)|$. By Fact 3, we have

$$\epsilon_q = \max_{k\in\mathbb{N}} |\hat{\mu}_{(A_q,\gamma)}(k) - \hat{\mu}_{(A'_q,\gamma')}(k)| =$$

$$\max_{k\in\mathbb{N}} |\int z^k d(\mu_{(A_q,\gamma)} - \mu_{(A'_q,\gamma')})| \le ||\mu_{(A_q,\gamma)} - \mu_{(A'_q,\gamma')}||.$$

By the second statement of Fact 5, we obtain $\epsilon_q \to 0$. On the other hand, using Lemma 1 for $B = \gamma(B_q)$ and $A = \gamma(A_q)$ (then $B \times A = \gamma(A_{q-1})$) and $t = 0$, we have, for every $0 \le s < n_q$,

$$\Phi_{\gamma(A_{q-1})}(s) = \Phi_{\gamma(B_q)}(s) + \Phi_{\gamma(B_q)}(n_q - s)\Phi_{\gamma(A_q)}(1), \text{ and}$$
$$\Phi_{\gamma(A_{q-1})}(n_q - s) = \Phi_{\gamma(B_q)}(n_q - s) + \Phi_{\gamma(B_q)}(s)\Phi_{\gamma(A_q)}(1),$$

from which we obtain

$$\Phi_{\gamma(B_q)}(s) = \frac{\Phi_{\gamma(A_{q-1})}(s) - \Phi_{\gamma(A_q)}(1)\Phi_{\gamma(A_{q-1})}(n_q - s)}{1 - \Phi^2_{\gamma(A_q)}(1)}$$

(by the observation following the statement of Fact 1, for large q the denominator is bounded away from zero). An analogous formula holds for γ', B'_q, A'_{q-1} and A'_q. Then

$$|\Phi_{\gamma(B_q)}(s) - \Phi_{\gamma'(B'_q)}(s)| \le \eta_q,$$

where η_q is a converging to zero function of $\epsilon_{q-1}, \epsilon_q, \xi_A$ and ξ'_A (not depending on s). This ends the proof. □

Theorem 3. *Suppose, with the assumptions of Fact 1, that the lengths $n_q = n'_q$ are bounded. Then $\mu_{(A,\gamma)}$ and $\mu_{(A',\gamma')}$ are equivalent for some γ and γ' if and only if there exists $q_0 \in \mathbb{N}$ such that*

$$\Phi_{\gamma(B_q)}(s) = \Phi_{\gamma'(B'_q)}(s),$$

for each $0 \le s < n_q$ and $q \ge q_0$.

Proof. Sufficiency follows from Fact 3 and Fact 4. Necessity is an immediate consequence of Theorem 2 and the observation that now there are only finitely many words to choose from. □

Remark 3. If the assertion of Theorem 3 holds then, by Lemma 1, it is seen that $\Phi_{\gamma(A_q)} = \Phi_{\gamma'(A'_q)}$, for q large enough, and hence, by Fact 3, $\mu_{(A_q,\gamma)} = \mu_{(A'_q,\gamma')}$.

Combining Theorem 1 with Theorem 3 we obtain the following characterization of spectral isomorphisms between Morse flows with bounded lengths of defining blocks:

Theorem 4. *Let A and A' be two Morse sequences over G defined by two sequences of blocks (B_q) and (B'_q), respectively, with the same structure of bounded lengths ($n_q = n'_q < M$ for each q). Then the Morse flows defined by A and A' are spectrally isomorphic if and only if there exists $q_0 \in \mathbb{N}$ and a permutation π of $\hat{G}$ such that $\pi(\gamma_0) = \gamma_0$, $\pi(\gamma^{-1}) = (\pi(\gamma))^{-1}$, and,*

$$\Phi_{\gamma(B_q)} = \Phi_{\pi(\gamma)(B'_q)},$$

for each $\gamma \in \hat{G}$, and $q \geq q_0$.

Proof. Only the property $\pi(\gamma^{-1}) = (\pi(\gamma))^{-1}$ needs a comment. Obviously, we have $\Phi_{\gamma^{-1}(B_q)} = \overline{\Phi_{\gamma(B_q)}}$. Thus if γ and $\pi(\gamma)$ satisfies the displayed formula, then the same holds for γ^{-1} and $(\pi(\gamma))^{-1}$. It is now not hard to see that the permutation can be modified to one that satisfies the required property. $\square$

Remark 4. The question arises: what are the possible pairs of words starting with 1 and having the same autocorrelation functions. Reconstructing a signal from its autocorrelation function is a subject in the field of Information Theory, unfortunately the "signal" has usually a slightly different setup than our "word". There are three natural cases, where two words, say, B and B' have the same autocorrelations:
(a) $B = (b_0, b_1, ..., b_{n-1})$ and B' is the "flipped word" $B^* = (b^*_0, b^*_1, ..., b^*_{n-1})$, where $b^*_i = \overline{b_{n-1-i}} b_n$,
(b) $B = C \times D$ and $B' = C \times D^*$,
(c) B is a concatenation of products: $B = (C_1 \times D)(C_2 \times D) \dots (C_k \times D)$, where $C_1, C_2, \dots C_k$ have the same length, and $B' = (C_1 \times D^*)(C_2 \times D^*) \dots (C_k \times D^*)$.
However, using a simple computer program, we have found pairs B, B' not being flip of each other and whose length is prime (which eliminates any product representation like in (b) or (c)). We classify such pairs as (d). The shortest such examples are over $\mathbb{Z}_3$ and have length 13:
$B = (1, 1, 1, 1, \bar{p}, \bar{p}, 1, 1, p, 1, p, \bar{p}, 1)$
$B' = (1, 1, 1, \bar{p}, \bar{p}, 1, p, p, p, \bar{p}, p, \bar{p}, 1)$,
and onother pair:
$C = (1, p, 1, 1, \bar{p}, 1, p, 1, \bar{p}, 1, 1, p, 1)$
$C' = (1, p, 1, \bar{p}, \bar{p}, p, \bar{p}, p, \bar{p}, \bar{p}, 1, p, 1)$
(p denotes the primary root of unity of degree 3). The second pair is more interesting because it does not satisfy the condition SA (see Definition 5 below). There also exists one example over $\mathbb{Z}_2$ of length 17.

Spectral isomorphisms governed by a group automorphism

Our main interest is in determining sufficient conditions forcing the permutation of Theorem 4 to be a group automorphism of $\hat{G}$. Of course, if the spectrum is not simple, then there are many permutations satisfying the assertion. In this case we are interested to find out whether among all such permutations there exists at least one group automorphism.

Recall that every group automorphism $\hat{v} : \hat{G} \to \hat{G}$ has the form

$$\hat{v}(\gamma)(g) = \gamma(v(g))$$

($g \in G$), where v is a group automorphism of G.

The case of $\mathbb{Z}_p$ for p prime

Consider the case where G is the cyclic group of roots of unity, denoted by $\mathbb{Z}_p$. First observe that if $p \leq 4$ then any permutation preserving the unity and inverse is an automorphism, and hence every spectral isomorphism between Morse flows over $\mathbb{Z}_p$ is governed by a group automorphism. But there are stronger reasons why the same must hold olso for larger prime numbers p.

Theorem 5. *Let $\mathbb{Z}_p$ be the (additive) cyclic group of order p where $p > 2$ is prime. Consider two Morse sequences A, A' over $\mathbb{Z}_p$ defined by two sequences of blocks (B_q) and (B'_q), with the same structure of bounded lengths. Then the corresponding Morse flows are spectrally isomorphic if and only if there exists a group automorphism $v : \hat{\mathbb{Z}}_p \to \hat{\mathbb{Z}}_p$ such that, for each $\gamma \in \hat{\mathbb{Z}}_p$, the spectral measures $\mu_{(A,\gamma)}$ and $\mu_{(A',v(\gamma))}$ are equivalent. Moreover, possible are only two cases*
(A) there are infinitely many numbers q for which the blocks B_q and B'_q are not symmetric; then both flows have simple spectrum and the automorphism v is unique,
(B) all blocks B_q and B'_q for $q \geq q_0$ are symmetric; then both flows have spectral multiplicity 2 (except for the discrete part) and there are two such automorphisms, v and $-v$.

Proof. Note that $\hat{\mathbb{Z}}_p = \mathbb{Z}_p$, and each character γ on $\mathbb{Z}_p$ has the form $\gamma_s(r) = \epsilon^{sr}$, where ϵ denotes the primary root of unity of degree p, and $r, s \in \mathbb{Z}_p$. Theorem 4 says that

$$\Phi_{\gamma_s(B_q)} = \Phi_{\gamma_{\pi(s)}(B'_q)},$$

for $q \geq q_0$. In this notation we have $\pi(0) = 0$ and $\pi(-s) = -\pi(s)$. We can also assume that $\pi(1) = 1$ (There exists a group automorphism $w : \mathbb{Z}_p \to \mathbb{Z}_p$ sending 1 to $\pi(1)$. The Morse sequence $w(A')$ is spectrally isomorphic to A via the permutation $\pi'(s) = w^{-1}(\pi(s))$, which clearly satisfies $\pi'(1) = 1$).

Fix $q \geq q_0$ and let n be the length of B_q. Applying the second part of Definition 2, for each $1 \leq k < n$ and $s \in \mathbb{Z}_p$ we can write

$$\Phi_{\gamma_s(B_q)}(k) = \sum_{r \in \mathbb{Z}_p} \epsilon^{sr} \mathrm{fr}_{B_q}(k, r),$$
$$\Phi_{\gamma_s(B'_q)}(k) = \sum_{r \in \mathbb{Z}_p} \epsilon^{sr} \mathrm{fr}_{B'_q}(k, r).$$

We will prove that

$$\mathrm{fr}_{B_q}(k, r) = \mathrm{fr}_{B'_q}(k, r)$$

for every $r \in \mathbb{Z}_p$. Consider the following polynomial of degree $p - 1$ with rational coefficients:

$$W(z) = \sum_{0 \leq r \leq p-1} z^r (\mathrm{fr}_{B_q}(k, r) - \mathrm{fr}_{B'_q}(k, r)).$$

Since $\pi(1) = 1$, we see that ϵ is one of zeros of this polynomial. On the other hand, $W(z)$ has a rational zero at 1 (because in each block the sum of frequencies eqals to 1). This yields that either $W(z) \equiv 0$ or ϵ is an algebraic number of degree $p - 2$. The last possibility contradicts the well known fact that ϵ is an algebraic number of degree $p - 1$.

Having established equality of corresponding frequencies in B_q and B'_q, we now see that $\Phi_{\gamma_s(B_q)}(k) = \Phi_{\gamma_s(B'_q)}(k)$, for every s, which implies that the identity permutation $\pi = id$ (which obviously is a group automorphism) governs the spectral isomorphism.

In order to prove the second part of the theorem, first assume that the identity is a unique such permutation. Then we conclude that for infinitely many indices $q \geq q_0$ both blocks B_q, B'_q are asymetric (if for each $q \geq q_0$ at least one of the blocks B_q, B'_q is symmetric then this block, say B, has the same autocorrelations as $-B$, hence the permutation $-id$ provides a second possibility.) Moreover, both flows then have simple spectrum, because otherwise there would be again more than one permutation. Thus we have the situation as in the assertion (A).

Suppose there exists another permutation π governing the spectral isomorphism. Let $B_q = (\epsilon^{r_0}, \epsilon^{r_1}, \dots, \epsilon^{r_{n-1}})$. For each s we have $\Phi_{\gamma_s(B_q)} = \Phi_{\gamma_{\pi(s)}(B'_q)}$, but also, as proved before, $\Phi_{\gamma_s(B_q)} = \Phi_{\gamma_s(B'_q)}$. Hence, we obtain

$$\Phi_{\gamma_s(B_q)} = \Phi_{\gamma_{\pi(s)}(B_q)}.$$

Let k be the smallest index for which at least one of the exponents r_k, r_{n-1-k} is different from 0. For simplicity of notation, we denote $a = r_k, b = r_{n-1-k}$. We have

$$\Phi_{\gamma_s(B_q)}(n-1-k) = \frac{1}{n}(z^{bs} + z^{-as}),$$
$$\Phi_{\gamma_{\pi(s)}(B_q)}(n-1-k) = \frac{1}{n}(z^{b\pi(s)} + z^{-a\pi(s)}).$$

We use the following elementary fact concerning unimodular complex numbers: if $z_1 + z_2 = z_3 + z_4 \neq 0$ then either ($z_1 = z_3$ and $z_2 = z_4$) or ($z_1 = z_4$ and $z_2 = z_3$). Since in $\mathbb{Z}_p$ no two numbers add to zero, we must have either

$$bs = b\pi(s) \text{ and } as = a\pi(s)$$
$$\text{or}$$
$$bs = -a\pi(s) \text{ and } as = -b\pi(s).$$

Since either a or b is nonzero, we have $\pi(s) = s$ in the first case, and $(\pi(s))^2 = s^2$ in the second. The last equation has in $\mathbb{Z}_p$ two solutions $\pi(s) = \pm s$.

Since $\pi \neq id$, $\pi(s) = -s$ is valid for some $s \neq 0$. Then we have

$$\Phi_{\gamma_s(B_q)} = \Phi_{\gamma_{-s}(B_q)}.$$

We will prove that B_q is symmetric. Suppose the converse, and let k be the smallest index for which $a = r_k \neq b = r_{n-1-k}$. We have

$$\Phi_{\gamma_s(B_q)}(n-1-k) = \frac{1}{n}(z^{bs} + R + z^{-as}),$$
$$\Phi_{\gamma_{-s})(B_q)}(n-1-k) = \frac{1}{n}(z^{-bs} + \overline{R} + z^{as}),$$

where R denotes the part of the formula involving the remaining (outer) terms of the block. But, by symmetry of the outer part of B_q, R is a real number, hence, as before, either

$$bs = -bs \text{ and } as = -as$$
$$\textit{or}$$
$$bs = as \text{ and } -as = -bs.$$

In either case $a = b$, a contradiction.

The fact that in the symmetric case the spectral multiplicity is not larger than 2 is now obvious: in the converse case there would exist a third permutation. This completes the proof of (B). □

Other cases

In the general case we can prove an appropriate positive theorem with some additional assumptions on the structure of the defining blocks (B_q) (we need them to be non-symmetric in a stronger sense). First we need the following observations:

Let π be any permutation of $\hat{G}$. Then π can be viewed as a Haar measure preserving transformation on $\hat{G}$. Thus it induces a unitary operator on $L^2(\hat{G})$. The characters on $\hat{G}$ form an orthogonal base in $L^2(\hat{G})$ and they have the form $\hat{g}(\gamma) = \gamma(g)$ $(g \in G)$. Developing their images $\hat{g} \circ \pi$ in the base we obtain:

$$(*) \qquad \hat{g} \circ \pi = \sum_{h \in G} \Pi(h, g)\hat{h},$$

where Π is some complex square matrix with rows and columns indexed by the elements of G.

Lemma 2. *Let π be the permutation of $\hat{G}$ as in Theorem 4. The matrix Π has the following properties:*
(i) Π is unitary,
(ii) $\Pi(h, g_1 g_2) = \sum_{e \in G} \Pi(e, g_1)\Pi(he^{-1}, g_2)$ $(h, g_1, g_2 \in G)$,
(iii) $\Pi(\mathbf{1}, \mathbf{1}) = 1$, $\Pi(\mathbf{1}, h) = \Pi(h, \mathbf{1}) = 0$ if $h \neq \mathbf{1}$,
(iv) $\Pi(h, g^{-1}) = \Pi(h^{-1}, g)$ $(h, g \in G)$

Proof. The properties (i) – (iii) follow from the fact that π represents a unitary and multiplicative operator on $L^2(\hat{G})$. The statements (iii) and (iv) can be easily derived from the conditions $\pi(\gamma_0) = \gamma_0$ and $\pi(\gamma^{-1}) = (\pi(\gamma))^{-1}$. We omit the detailed calculations. □

Lemma 3. *Let A and A' be as in Theorem 4 and suppose the corresponding Morse flows are spectrally isomorphic. Then for every $q \geq q_0$, $0 \leq k < n_q$ and $h \in G$ we have*

$$\mathrm{fr}_{B'_q}(k, g) = \sum_{h \in G} \overline{\Pi(h, g)}\ \mathrm{fr}_{B_q}(k, h).$$

Proof. Denote for short $B = B_q$, $B' = B'_q$ and $n = n_q$. By Theorem 4, we have

$$\Phi_{\gamma(B)} = \Phi_{\pi(\gamma)(B')},$$

i.e., for every fixed $0 \leq k < n$,

$$\sum_{h \in G} \gamma(h)\ \mathrm{fr}_B(k, h) = \sum_{g \in G} \pi(\gamma)(g)\ \mathrm{fr}_{B'}(k, g).$$

The right hand side can be rewritten as $\sum_{g \in G}(\hat{g} \circ \pi)(\gamma)\ \mathrm{fr}_{B'}(k, g)$. By the formula (*) it then becomes

$$\sum_{g \in G} \sum_{h \in G} \Pi(h, g)\hat{h}(\gamma)\ \mathrm{fr}_{B'}(k, g) = \sum_{h \in G} \gamma(h) \sum_{g \in G} \Pi(h, g)\ \mathrm{fr}_{B'}(k, g)$$

(the equalities hold for each $\gamma \in \hat{G}$). Viewing the above as a function of the variable $\gamma \in \hat{G}$ and by uniqueness of the representation in the basis $\{\hat{h} : h \in G\}$ we obtain $\mathrm{fr}_B(k,h) = \sum_{g\in G} \Pi(h,g)\ \mathrm{fr}_{B'}(k,g)$ for each h. Since Π is unitary, $\Pi^{-1} = \overline{\Pi^T}$, and the desired equality holds. $\square$

Suppose that $\Pi(h,g) = 1$ for some $h, g \in G$. Then, by (i), $\Pi(e,g) = 0$ if $e \neq h$ and $\Pi(h,e) = 0$ if $e \neq g$.

Definition 4. We say that $g \in G$ has the *permutation property* if $\Pi(h,g) = 1$ for some $h \in G$. Then we denote by $v(g)$ the so determined element h.

Lemma 4. *The set G_v of those $g \in G$ which have the permutation property is a subgroup, and v is a group isomorphism between G_v and its image by v.*

Proof. By (iii) $\mathbf{1} \in G_v$, and by (iv) $g \in G_v \implies g^{-1} \in G_v$. It suffices to check that $g_1, g_2 \in G_v \implies g_1g_2 \in G_v$ and $v(g_1g_2) = v(g_1)v(g_2)$. Indeed, by (ii)

$$\Pi(h, g_1g_2) = \sum_{e\in G} \Pi(e, g_1)\Pi(he^{-1}, g_2).$$

By the permutation property of g_1 and g_2, the only nonzero summand is where $e = u(g_1)$ and simultaneously $he^{-1} = v(g_2)$. Since nonzero summand exists for some h, we obtain $h = v(g_1)v(g_2)$. By uniqueness of such h, $\Pi(h, g_1g_2) = 1$, hence $g_1g_2 \in G_v$ and $v(g_1g_2) = v(g_1)v(g_2)$, as needed. $\square$

We will prove the group automorphism property of spectral isomorphism between Morse sequences whose defining blocks B_p satisfy certain condition of asymmetry. For a given block $B = (b_0, b_1, \dots, b_{n-1}) \in G^n$ and $0 \le k \le n/2$ we denote by G^B_k the subgroup of G generated by the elements $b_0, b_1, \dots, b_{k-1}, b_k$ and their symmetric correspondents $b_{n-1}, b_{n-2}, \dots, b_{n-k}, b_{n-k-1}$. If $k > n/2$ then we put $G^B_k = G^B_{k-1}$.

Definition 5. We say that a sequence of blocks (B_q) *has property AS* if for each $k \in \mathbb{N}$ and $q \in \mathbb{N}$ at least one of the elements b_k, b_{n-k-1} of B_q belongs to the group G_{k-1} generated by $\bigcup_q G^{B_q}_{k-1}$.

Remark 6. The above class of blocks is quite large. For example, in cyclic groups it suffices that the last term of the block is a generator.

Theorem 6. *Let A and A' be a pair of spectrally isomorphic Morse sequences over G, with the same structure of bounded lengths. The number q_0 is thus given (see Theorem 4). If $(B'_q)_{q\ge q_0}$, (or $(B_q)_{q\ge q_0}$) has property AS then π is a group automorphism.*

Proof. Suppose $(B'_q)_{q\ge q_0}$ has property AS. We will inductively prove that $G'_k \subset G_v$ and that $v(G'_k) = G_k$. Since $\bigcup_{k\in\mathbb{N}} G'_k = G$, the automorphism v will eventually extend to the whole group G.

Step 0

Fix $q \ge q_0$ and denote $B = B_q$, $B' = B'_q$ and $n = n_q$. By Theorem 4, for every $\gamma \in \hat{G}$ we have $\Phi_{\gamma(B)}(n-1) = \Phi_{\pi(\gamma)(B')}(n-1)$. By the definition of autocorrelations this simply means that

$$\gamma(b_{n-1}) = \pi(\gamma)(b'_{n-1}),\ \text{for each } \gamma,$$

i.e., that $\hat{b}'_{n-1} \circ \pi = \hat{b}_{n-1}$. By the formula (*) we obtain that

$$\Pi(b_{n-1}, b'_{n-1}) = 1.$$

We have proved that b'_{n-1} has the permutation property, and $v(b'_{n-1}) = b_{n-1}$. Recall that $b'_0 = b_0 = \mathbf{1}$, hence $v(b'_0) = b_0$. We do so for each $q \geq q_0$. The application of Lemma 4 yields $G'_0 \subset G_v$ and $v(G'_0) = G_0$.

Step k

Suppose the statement has been proved for $k-1$. As before, fix $q \geq q_0$ and denote $B = B_q$, $B' = B'_q$ and $n = n_q$. For every $\gamma \in \hat{G}$ we have

$$\Phi_{\gamma(B)}(n-k-1) = \Phi_{\pi(\gamma)(B')}(n-k-1),$$

i.e.,

$$\sum_{g \in G} \gamma(g)\ \mathrm{fr}_B(n-k-1, g) = \sum_{g \in G} \pi(\gamma)(g)\ \mathrm{fr}_{B'}(n-k-1, g).$$

The above can be written as

$$L = \sum_{g \in G_{k-1}} \gamma(g)\ \mathrm{fr}_B(n-k-1, g) + \sum_{g \notin G_{k-1}} \gamma(g)\ \mathrm{fr}_B(n-k-1, g) =$$
$$\sum_{g \in G'_{k-1}} \pi(\gamma)(g)\ \mathrm{fr}_{B'}(n-k-1, g) + \sum_{g \notin G'_{k-1}} \pi(\gamma)(g)\ \mathrm{fr}_{B'}(n-k-1, g) = R.$$

Note that if $g \in G_v$ then, by (*), $\hat{g} \circ \pi = \widehat{v(g)}$, and, by Lemma 3, $\mathrm{fr}_{B'}(k, g) = \mathrm{fr}_B(k, v(g))$. Thus the first sum of R becomes

$$\sum_{g \in G'_{k-1}} \gamma(v(g))\ \mathrm{fr}_B(n-k-1, v(g)),$$

which, by the assumption that $v(G'_{k-1}) = G_{k-1}$ equals to the first sum of L. By property AS, the second sum of R consists of at most one summand: either $\frac{1}{n}\pi(\gamma)(b'_{n-k-1})$ or $\frac{1}{n}\pi(\gamma)({b'_k}^{-1} b'_{n-1})$; this is best seen if R is written as

$$\frac{1}{n}\pi(\gamma)(b'_{n-k-1}) + \frac{1}{n}\sum_{i=1}^{k-1} \pi(\gamma)({b'_i}^{-1} b'_{i+n-k-1}) + \frac{1}{n}\pi(\gamma)({b'_k}^{-1} b'_{n-1}),$$

because all elements ${b'_i}^{-1} b'_{i+n-k-1}$ in the central sum belong by definition to G'_{k-1}. Letting $\gamma = \gamma_0$ in the expressions L an R, their first sums become the sums of the corresponding frequencies over G'_{k-1} and G_{k-1}, so, by the previous argument, their common value is either 1 or $1 - \frac{1}{n}$. It is now seen that the second sum of L has as many summands as that of R, i.e., zero or one. The case of zero summands is trivial. Suppose we have one summand on each side, say $\frac{1}{n}\pi(\gamma)(g)$ and $\frac{1}{n}\gamma(h)$. Then $\pi(\gamma)(g) = \gamma(h)$ for each γ. As before, by the formula (*) we obtain $\Pi(h, g) = 1$. We have proved that $g \in G_v$, and $v(g) = h$. We do so for each $q \geq q_0$. So obtained elements g_q exhaust all new elements generating G'_k and G_k, ("new" means not in G'_{k-1}), and h_q exhaust all new elements generating G_k. The application of Lemma 4 yields $G'_k \subset G_v$ and $v(G'_k) = G_k$. $\square$

References

[C-N] J. R. Choksi & M .G. Nadkarni, *The maximal spectral type of a rank one transformation*, Canad. Math. Bull. **37** (1994), 29–36.

[D-L] T. Downarowicz & Y. Lacroix, *Merit factors and Morse sequences*, Theoretical Computer Science (to appear).

[Gu] M. Guenais, *Morse cocycles and simple Lebesgue spectrum*, to appear, Ergodic Th. & Dyn. Syst. (1998).

[J] A. del Junco, *A transformation with simple spectrum which is not rank one*, Can. J. Math. **29** (1977), 653-663.

[J-L-M] A. del Junco, M. Lemanczyk & M. Mentzen, *Semisimplicity, joinings and group extensions*, Studia Math. **112** (1995), 141-164.

[I-L] A. Iwanik & Y. Lacroix, *Some constructions of strictly ergodic non-regular Toeplitz flows*, Studia Math. **110** (1994), 191–203.

[K1] M. S. Keane, *Generalized Morse sequences*, Zeit. Wahr. Verw. Geb. **10** (1968), 335–353.

[K2] M. S. Keane, *Strongly mixing g-measures*, Invent. Math. **16** (1972), 309–324.

[K-S] J. Kwiatkowski & A. Sikorski, *Spectral properties of G-symbolic Morse shifts*, Bull. Soc. Math. France **115** (1987), 19-33.

[M] J. C. Martin, *The structure of generalized Morse minimal sets on n-symbols*, P.A.M.S. **2** (1977), 343-355.

[N] D. Newton, *On canonical factors of ergodic dynamical systems*, J. London Math. Soc. **19** (1979), 129-136.

Institute of Mathematics, Technical University, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland.
E-mail address: `downar@im.pwr.wroc.pl`

Faculty of Mathematics and Informatics, Nicholas Copernicus University, Chopina 12/18, 87-100 Toruń, Poland.
E-mail address: `jkwiat@mat.uni.torun.pl`

U. B. O., Fac. des Sciences et Techniques, Département de Maths, 6 Av. V. Le Gorgeu, B.P. 809, 29285 Brest Cedex, France.
E-mail address: `lacroix@univ-brest.fr`