A CRITERION FOR TOEPLITZ FLOWS TO BE TOPOLOGICALLY ISOMORPHIC AND APPLICATIONS

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Introduction

A dynamical system is said to be coalescent if its only endomorphisms are automorphisms. The question whether there exist coalescent ergodic dynamical systems with positive entropy has not been solved so far and it seems to be difficult. The analogous problem in topological dynamics has been solved by Walters ([W]). His example, however, is not minimal. In [B-K2], a class of strictly ergodic (hence minimal) Toeplitz flows is presented, which have positive entropy and trivial topological centralizers (the last condition implies coalescence). The entropy, however, is only estimated from below. Also the class is obtained in a not completely constructive way.

The basic idea of this paper is contained in Section 2, in a criterion which describes homomorphisms (isomorphisms) between Toeplitz flows in terms of a block code simplified to a function sending blocks of a given length to blocks of the same length. This idea is then applied in Section 3 to effectively construct an uncountable family of pairwise nonisomorphic Toeplitz flows with topological entropy equal to a common arbitrarily preset value. Furthermore, all the Toeplitz flows have the same maximal uniformly continuous factor.

In Section 4 we obtain conditions sufficient for coalescence of a Toeplitz flow. In particular, all Toeplitz flows of Section 3 turn out to be coalescent.

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1. Preliminaries

We shall use $\mathbf{Z}$ and $\mathbf{N}$ to denote the set of all integers and the set of all positive integers, respectively. By a *flow* we mean a pair (X,T), where X is a compact metric space and T is a homeomorphism of X. Recall that a flow (X,T) is said to be *minimal* provided X contains no proper nonempty T-invariant closed subsets. (X,T) is said to be *uniquely ergodic* if there exists a unique T-invariant Borel probability measure on X. We say that (X,T) is *strictly ergodic* if it is both minimal and uniquely ergodic.

By the *orbit-closure* of an element $x \in X$ we mean the set

$$\overline{O}(x) = \overline{\{T^n(x) : n \in \mathbf{Z}\}}.$$

X is minimal if and only if it is equal to every orbit-closure $\overline{O}(x)$, with $x \in X$.

Let (Y,S) be another flow and let $\pi : X \mapsto Y$ be an onto continuous mapping such that $\pi \circ T = S \circ \pi$. We then say that (Y,S) is a *factor* of (X,T) (or (X,T) is an *extension* of (Y,S)).The mapping π

is then called a *homomorphism* from the flow (X,T) to the flow (Y,S). The flows (X,T) and (Y,S) are said to be *topologically isomorphic* provided there exists a bijective homomorphism from (X,T) to (Y,S).

The *topological centralizer* of a flow (X,T), denoted by $C_{top}(X,T)$, is the set of all homomorphisms from (X,T) to itself (endomorphisms). A flow (X,T) is called *coalescent* if $C_{top}(X,T)$ consists exclusively of isomorphisms (automorphisms). An even more restricted case occurs when $C_{top}(X,T)$ is *trivial*, i.e., when it equals $\{T^n : n \in \mathbf{Z}\}$.

Every minimal flow admits a *maximal uniformly continuous factor*, $(G,1)$, where G is a compact monothetic group and 1 denotes the rotation by a generator 1 of G.

All considerations of this note will be devoted to the case when X is a compact subset of $\Sigma^{\mathbf{Z}}$, where Σ is a finite set called an *alphabet*, and $T = S$ is the left shift transformation. An element

$$x = (\dots, x(-2), x(-1), x(0), x(1), x(2), \dots)$$

of X will be viewed as a two-sided sequence over the alphabet Σ, while the indices will be called *positions* or *coordinates*. The position 0 is frequently called the *central position*.

By a *block* B over Σ we shall mean a finite sequence of elements of Σ, and by $|B|$ we will denote its length. If $x \in X$ then by $x[m,n)$ we mean the block

$$x[m,n) = (x(m), x(m+1), \dots, x(n-1)).$$

We say that a block B *appears* in x at the mth position if $B = x[m, m+|B|)$. We say that a block $x[m,n)$ appears *periodically* in x if there exists a positive integer p (a period) such that $x[m,n)$ appears in x at the positions $(m+kp)$ for all integers k.

Each homomorphism π of flows which are subshifts over a finite alphabet Σ is determined by so called *block code*, i.e., a function $C : \Sigma^{2n} \mapsto \Sigma$ ($n \in \mathbf{N}$ will be called the length of the code) in the following way:

$$\pi x(m) = C(x[m-n, m+n)).$$

Toeplitz sequences over Σ are those in which every block appears periodically. Usually we exclude periodic sequences. If ω is a Toeplitz sequence we can select the following objects (see [Wi]):

1) *p-periodic parts*, $\mathrm{Per}_p(\omega) = \{n \in \mathbf{Z} : \omega(n)$ appears in ω p-periodically$\}$. A period p is called *essential* if it is the smallest of the periods which yield the same periodic part.

2) a *period structure* of ω, which is an increasing sequence $p_0, p_1, p_2, \dots$ of essential periods with the following properties: the period of any block in ω is a multiple of some p_t, $p_0 \geq 2$, and for each nonnegative integer t, p_t divides p_{t+1}.

3) *t-skeleton* (for each t), which equals $\mathrm{Per}_{p_t}(\omega)$. The positions in the complement to the t-skeleton will be called *t-holes*. Intervals (possibly empty) contained between two consecutive t-holes will be called *t-intervals*.

4) the collection $W_t(\omega)$ (for each t) of all blocks of the form $\omega[kp_t, (k+1)p_t)$ called *t-symbols*.

Note that t-symbols from $W_t(\omega)$ may differ only at the t-holes. From the definition of the Toeplitz sequence it follows that for a given integer n there exists t such that the t-symbols have t-holes at least n positions away from both the ends. (Pick t with p_t equal to a multiple of the period of $\omega[-n,n)$).

It is known that every Toeplitz flow $(\overline{O}(\omega), S)$ is minimal, and that the maximal uniformly continuous factor of $(\overline{O}(\omega), S)$ is identical with the p-addic integers $(G_p, 1)$, where p is a period structure (p_t) (See [Wi]). It is worth noticing that the corresponding homomorphism from $\overline{O}(\omega)$ to G_p is 1-1 exactly on the subset of $\overline{O}(\omega)$ consisting of Toeplitz sequences.

The monothetic group G_p consists of elements of the form

$$h = (h(t)) \in \prod_{t=1}^{\infty} \{0, 1, \dots, p_t - 1\}$$

such that $h(t+1) = h(t) \bmod (p_t)$. By identifying every such element $h \in G_p$ which is constantly j starting at some t_0, with the positive integer j, and every such h for which $h(0) = p_t - j$ starting at some t_0, with the negative integer $-j$, the group G_p can be viewed as a compactification of the integers. In this setting $1 = (1, 1, 1, \dots)$ is the generator of G_p. The sets

$$H_t = \{h \in G_p : h(s) = 0 \text{ for } s \leq t\}$$

are closed and open subgroups admitting exactly p_t cosets:

$$H_t, H_t + 1, H_t + 2, \dots, H_t + p_t - 1.$$

The cosets of the subgroups H_t form a base for the topology in G_p. In this notation every function $f : G_p \mapsto \Sigma$ defines a sequence over Σ, by simply restricting the domain to $\mathbf{Z}$.

A function $f : G_p \mapsto \Sigma$ defines a Toeplitz sequence if f is continuous at each integer $j \in G_p$ (i.e., if for each j there exists a t such that f is constant on the coset of H_t which contains j). As in [D1], it is necessary to only consider functions f with unremovable discontinuities, i.e., such that the set of discontinuities of f, $D(f)$, cannot be made smaller by only changing the function on $D(f)$. We denote the family of such functions by F. If $f \in F$ then $D(f)$ is nowhere dense and closed. Every Toeplitz sequence ω is defined by a function $f : G_p \mapsto \Sigma$, $f \in F$, where G_p is the maximal uniformly continuous factor of $(\overline{O}(\omega), S)$ (see [D1], [D-I]).

Given a function $f : G_p \mapsto \Sigma$ and $t \geq 0$, we can define a function $f^{(t)} : H_t \mapsto \Sigma^{p_t}$ by the formula

$$f^{(t)}(h) = (f(h), f(h+1), \dots, f(h + p_t - 1)), \quad (h \in H_t).$$

Note that

$$D(f^{(t)}) = \bigcup_{j=0}^{p_t - 1} (D(f) \cap (H_t + j)) - j.$$

If $f \in F$ and $\mathbf{Z} \cap D(f) = \emptyset$ then the same holds for $f^{(t)}$ (the domain of $f^{(t)}$, H_t, can also be viewed as a group of the form G_p). It is not hard to see that if now $h \notin D(f^{(t)})$ then $f^{(t)}(h) \in W_t(\omega)$, where ω is the Toeplitz sequence defined by f.

2. Isomorphism theorems

Let ω and η be two Toeplitz sequences over an alphabet Σ, and having the same period structure p. (Two Toeplitz sequence over finite alphabets can always be viewed as sequences over a common finite alphabet). Suppose there exists a homomorphism π from $(\overline{O}(\omega), S)$ to $(\overline{O}(\eta), S)$. Then π preserves the maximal uniformly continuous structures of both flows, so it induces a homomorphism π' from G_p to itself. But every such homomorphism is the multiplying by an element h_0 in G_p. Thus, denoting by π_1 and π_2 the natural homomorphisms from $(\overline{O}(\omega), S)$ and from $(\overline{O}(\eta), S)$ to G_p, respectively, we obtain the following commutative diagram:

$$\begin{array}{ccc} \overline{O}(\omega) & \xrightarrow{\pi} & \overline{O}(\eta) \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ G_p & \xrightarrow[h_0]{} & G_p \end{array}$$

We then say that π is *over* h_0. Notice that from the diagram and earlier remarks it follows that π sends Toeplitz sequences in a 1-1 way to Toeplitz sequences.

We first consider the existence of homomorphisms from $(\overline{O}(\omega), S)$ to $(\overline{O}(\eta), S)$ which are over zero. It is obvious that an equivalent condition for such homomorphism to be over zero is that $\pi\omega = \eta$, because ω and η are the unique elements sent to zero by π_1 and π_2, respectively.

Theorem 1. *There exists a homomorphism (isomorphism) π from $(\overline{O}(\omega), S)$ to $(\overline{O}(\eta), S)$ over zero if and only if for some $t \geq 0$ there exists a function (bijective function) $\Pi : W_t(\omega) \mapsto W_t(\eta)$ such that*

$$\eta[kp_t, (k+1)p_t) = \Pi(\omega[kp_t, (k+1)p_t))$$

for every integer k.

Proof. Necessity will be derived from the fact that π is determined by some block code C. Namely, let t be such that the holes in the t-symbols of ω are at least n positions away from the ends, where n is the length of the code C. Note that now each t-symbol in ω is preceded and followed by at least n fixed symbols (not depending on the particular choice of the t-symbol), and hence it determines through C the entire corresponding t-symbol of η. Thus the function Π can be defined. For sufficiency notice the following: every $\omega' \in \overline{O}(\omega)$ is a concatenation of the t-symbols, perhaps shifted to positions of the form $kp_t + j$, for some $0 \leq j < p_t$. Thus the application of the function Π to ω' (in a natural way: t-symbol after t-symbol) will produce a concatenation of the t-symbols of η shifted to the same positions $kp_t + j$. This procedure defines a function π from $(\overline{O}(\omega), S)$ to some set of sequences. Now, π is easily seen to be continuous and shift-preserving, hence the image by π is a minimal subshift. But, since $\pi\omega = \eta$, the image must equal $\overline{O}(\eta)$, and π is a homomorphism from $(\overline{O}(\omega), S)$ to $(\overline{O}(\eta), S)$ as desired. The isomorphism case is now immediate. □

The above theorem can also be expressed in terms of the functions on G_p defining the Toeplitz sequences ω and η. The proof of the following statement in an immediate consequence of Theorem 1 along with the fact that if $f \in F$ then f restricted to $\mathbf{Z}$ fully determines f except on $D(f)$, and of the remarks following the definition of $f^{(t)}$.

Theorem 2. *Let $f : G_p \mapsto \Sigma$ and $g : G_p \mapsto \Sigma$ be such that $f \in F, g \in F$, $\mathbf{Z} \cap D(f) = \emptyset$, $\mathbf{Z} \cap D(g) = \emptyset$ i.e., both functions define Toeplitz sequences, say ω and η, respectively. Then there exists a homomorphism (isomorphism) over zero from $(\overline{O}(\omega), S)$ to $(\overline{O}(\eta), S)$ if and only if for some $t \geq 0$ there exists a function (bijective function) $\Pi : W_t(\omega) \mapsto W_t(\eta)$ such that for every $h \in H_t \setminus D(f^{(t)})$*

$$g^{(t)}(h) = \Pi(f^{(t)}(h)).$$

Note that if the above is satisfied then $D(g^{(t)}) \subset D(f^{(t)})$, and in the isomorphism case we have equality. Also note that the function Π is always onto.

Our next step is getting rid of the assumption that the homomorphism is over zero. An appropriate theorem will be formulated in terms of the functions on G_p .

Theorem 3. *With the assumptions as in Theorem 2, there exists a homomorphism (isomorphism) from $(\overline{O}(\omega), S)$ to $(\overline{O}(\eta), S)$ over some $h_0 \in G_p$ if and only if for some $t \geq 0$ there exists a function (1-1 function) $\Pi : W_t(\omega) \mapsto \Sigma^{p_t}$ such that for every $h \in H_t \setminus D(f^{(t)})$*

$$(g \circ h_0)^{(t)}(h) = \Pi(f^{(t)}(h)),$$

where $g \circ h_0$ denotes the ordinary composition of g with the multiplication by h_0 on G_p.

Proof. First observe that if $g \circ h$ defines a Toeplitz sequence, η' for an $h \in G_p$ then $\eta' \in \overline{O}(\eta)$ (see [D1]). Also η' is the unique element in the preimage of h by π_2, the maximal uniformly continuous factor of $(\overline{O}(\eta), S)$. Now, if a homomorphism π over h_0 exists then $\pi\omega$ is a Toeplitz sequence, and it is the one defined by $g \circ h_0$. Next apply Theorem 2 to ω and $\pi\omega$. Conversely, the formula involving Π in the assertion yields, by Theorem 2, the existence of a homomorphism over zero from $(\overline{O}(\omega), S)$ to the orbit-closure of the Toeplitz sequence, η', defined by $g \circ h_0$. Applying the rotation by $-h$ we obtain that η belongs to this orbit-closure, which hence equals $\overline{O}(\eta)$. Thus we have a homomorphism from $(\overline{O}(\omega), S)$ to $(\overline{O}(\eta), S)$ over h_0, since the last is the image of η' by π_2. $\square$

3. An uncountable family

We now present a construction of an uncountable family of Toeplitz sequences over two symbols with the following properties: the corresponding Toeplitz flows are strictly ergodic with zero entropy, they all have the same maximal uniformly continuous factor, and are pairwise nonisomorphic. Later we modify the construction to obtain another family of Toeplitz flows with the same properties except that the topological entropy will have an arbitrarily preset common value between 0 and $\log 2$. Below we will identify the group G_p with its homeomorphic copy (a Cantor set) $\mathbf{C}$ on an interval.

Construction. Let $\Sigma = \{0, 1\}$. Consider the quotient equivalence relation R of $G_p/\mathbf{Z}$ in G_p:

$$h_1 R h_2 \iff h_1 - h_2 \in \mathbf{Z}$$

Obviously, the corresponding equivalence classes are countable, hence the number of classes is continuum. Choose one representative c_α from each class except the class of zero (α indexes the classes). Next, for each α find $a_\alpha \notin \mathbf{Z}$ and $b_\alpha \notin \mathbf{Z}$ such that $b_\alpha - a_\alpha = c_\alpha$, and let $f_\alpha : G_p \mapsto \Sigma$ be equal to $1_{[a_\alpha, b_\alpha] \cap \mathbf{C}}$. Obviously, $f_\alpha \in F$. Since f_α has the only discontinuities at a_α and b_α, it defines a Toeplitz sequence, say η_α. It follows e.g. from [D-I] that η_α is regular, hence its orbit-closure is strictly ergodic and with entropy zero. It is not hard to see that the maximal uniformly continuous factor of the Toeplitz flow generated by the Toeplitz sequence η_α equals G_p (see e.g. [D1], Theorem 3). Note that for any $h \in G_p$ the function $f_\alpha \circ h$ has exactly two discontinuities differing by c_α. Thus, for a given $t \geq 0$, the difference between the only two discontinuity points of the function $(f_\alpha \circ h)^{(t)}$ is R-equivalent to c_α. Now, applying Theorem 3 and the note following Theorem 2, we obtain that the Toeplitz flows generated by the Toeplitz sequences η_α are pairwise nonisomorphic.

Finally, we apply the technique of "mixing" to modify the obtained family of Toeplitz flows to one with a common nonzero entropy. This technique has been already presented in [D-I] and then developed in [D2]. Pick a positive integer n and let $q_0 = n$ and $q_t = np_{t-1},\ (t > 0)$. The group G_q can be identified with a union of n disjoint copies of G_p on an interval. Choose a Toeplitz sequence η with the period structure q and topological entropy equal to a preset

value $\epsilon \in (0, \log 2)$. Then let $f : G_q \mapsto \Sigma$ be a function in F defining η. The existence of such objects is guaranteed in [D-I]. Next, let $r_0 = n + 1$ and $r_t = (n + 1)p_{t-1},\ (t > 0)$. The group G_r can be viewed as a disjoint union of one copy of G_p and one of G_q. For each index α define a function $g_\alpha : G_r \mapsto \Sigma$ by putting f_α on the copy of G_p (as in the first part of the construction) and f on the copy of G_q. It is seen that $g_\alpha \in F$ and it defines a Toeplitz sequence, say ω_α. (The sequence ω_α is obtained from η_α and η by setting their entries alternately: $\omega_\alpha((n + 1)m) = \eta_\alpha(m),\ \omega_\alpha((n + 1)m + k) = \eta((n)m + k - 1),\ (0 < k \leq n)$). Suppose there exists a homomorphism over some $h_0 \in G_r$ from the orbit-closure of one of the ω_α to another. If $h_0(0) = 0$ then h_0 sends the copy of G_p in G_r to itself, hence it induces a homomorphism of the orbit-closures of the corresponding sequences η_α, which is impossible. In other cases consider h^n, to obtain the same contradiction. Similar argument can be used to show that the maximal uniformly continuous factor for each α is G_r. As easily computed, the topological entropy of ω_α equals $\epsilon \frac{n}{n+1}$, which can be an arbitrary number in $(0, \log 2)$.

4. The problem of coalescence

The question whether all Toeplitz flows are coalescent seems to remain still open. We can give a positive answer for flows generated by several special types of Toeplitz sequences. Some of the results stated below can be found in earlier papers, however, applying the isomorphism criterion simplifies the proofs. The following idea covers most of previously obtained results.

Lemma 1. *Suppose π is a noninvertible endomorphism of a Toeplitz flow $(\overline{O}(\omega), S)$. Let n be the length of the corresponding block code. Then, for every $t \geq$ some t_0, and a t-interval F_0 of length $\geq 2n$ there exist t-intervals $F_1, F_2, \ldots, F_k$ with the following properties:*
1) $|F_0| = m_t = max\{|F| : F \text{ a } t\text{-interval}\}$
2) $|F_{i-1}| - 2n < |F_i| < |F_{i-1}|,$ *for* $i = 1, 2, \ldots, k$
3) $|F_k| < 2n.$

Proof. The proof will consist of three parts.
Part 1. First observe that all possible systems of t-symbols to be found as $W_t(\omega')$ with $\omega' \in \overline{O}(\omega)$, are those obtained by dividing up ω into blocks of length p_t (which is possible in exactly p_t different ways). Every such partition induces a partition of the t-skeleton and of the complementary set of t-holes (both are p_t-periodic sets). We say that such partition hits a t-interval F if it separates the two t-holes at the ends of F. Now, the number of different t-symbols obtained by one such partition of ω is determined by the induced partition of the set of t-holes. Hence, if two partitions hit the same t-interval, they result with equally rich sets of t-symbols.
Part 2. Imagine the Toeplitz sequences ω, $\pi\omega$, $\pi^2\omega, \ldots$ written one under another with the central positions lined up. Find in ω some t-interval F_0 with $|F_0| \geq 2n$. Clearly, there are no t-holes directly below the central part of F_0, $F_0[n, |F_0| - n)$, which means that below F_0 there appears a t-interval, say F_1, of length at least $|F_0| - 2n$. This can be repeated as long as the obtained t-intervals, F_1, $F_2, \ldots$ have lengths $\geq 2n$. This sequence satisfies $|F_{i-1}| - 2n < |F_i|$. In each step, say in the ith step, apply a partition of $\pi^{i-1}\omega$ hitting F_{i-1} in its center, i.e., so that the resulting t-symbols have t-holes at least n positions away from their ends. Now, as in the proof of Theorem 1, passing to underlying t-symbols in $\pi^i\omega$ coincides with the action of the function Π. Since π is noninvertible, there must be less different t-symbols below than above. The corresponding partition of $\pi^i\omega$ hits the t-interval F_i, possibly not in its center. If $|F_i| \geq 2n$ then the partition can be shifted to the center of F_i without changing the number of resulting t-symbols (see Part 1). We can now repeat the whole procedure in the next step. Since the number of t-symbols cannot decrease infinitely, it is seen that in some step we arrive to a t-interval with length $< 2n$, as desired in 3)
Part 3. Starting the above construction from a t-interval with maximal
length and then choosing a subsequence with appropriately decreasing lengths, we obtain a sequence as in the assertion. □

The following condition was introduced in [B-K1]:

Definition 1. We say that a Toeplitz sequence ω has *separated holes* if for each positive integer n there exists $t \geq 0$ such that every two consecutive t-holes in ω are at least $2n$ positions apart.

It is not hard to see that in term of the function $f : G_p \mapsto \Sigma$ defining a Toeplitz sequencethe above condition means that D_f has at most one representative in every R-equivalence class. We can now improve a result of [B-K1] where it was stated only for sequences over two symbols:

Theorem 4. *If ω is a Toeplitz sequence with separated holes then $(\overline{O}(\omega), S)$ is coalescent.*

Proof. For sequences with separated holes the condition 3) of Lemma 1 can not be satisfied for large t. □

It was proved in [B-K2] that the centralizer of a Toeplitz flow obtained from so called Oxtoby sequence is trivial (which implies coalescence of such flows). As an application of Lemma 1 we can derive coalescence for Oxtoby sequences almost immediately. First recall the definition of an Oxtoby sequence (cf. condition (*) in [B-K2]):

Definition 2. A Toeplitz sequence ω is called an *Oxtoby sequence* provided for each $t > 0$ and each integer k, if the interval $\omega[kp_t, (k+1)p_t)$ contains a $(t+1)$-hole then all t-holes in this interval are $(t+1)$-holes.

Now, for an Oxtoby sequence η, choose t such that $p_{t-1} > 4n$. As easily seen, if $|F_0| = m_t > 2p_{t-1}$, then the next available smaller length of a t-interval is shorter by at least p_{t-1}, hence a t-interval F_1 as in Lemma 2 cannot exist. Suppose that

$$2p_{t-1} > m_t > p_{t-1}$$

for all sufficiently large t. Then observe F_0, F_1 and F_2, the longest, second longest and third longest $(t+1)$-intervals. We have

$$|F_0| = m_{t+1} > p_t, \quad |F_1| \leq m_t < p_t \ \text{ and } \ |F_2| \leq m_{t-1} < p_{t-1}.$$

Since $p_t \geq 2p_{t-1}$, we obtain that

$$|F_0| - |F_2| > p_{t-1} > 4n.$$

Again, the sequence of Lemma 2 cannot exist, and the coalescence follows.

Finally, we can point out another condition sufficient for coalescence of a Toeplitz flow. Recall that a Toeplitz sequence is called *regular* if the densities in $\mathbf{Z}$ of the t-skeletons tend to 1, or, equivalently, if $\frac{h_t}{p_t} \to 0$, where h_t denotes the number of t-holes in each period. A condition slightly stronger than regularity is used in the following statement:

Theorem 5. *If*

$$\frac{h_t^2}{p_t} \underset{t}{\longrightarrow} 0$$

then the Toeplitz flow $(\overline{O}(\omega), S)$ is coalescent.

Proof. By Lemma 1, in noncoalescent case there appear at least $\frac{m_t}{2n}$ different t-intervals, hence also $h_t \geq \frac{m_t}{2n}$ (n a positive integer). On the other hand, $m_t \geq \frac{p_t}{h_t} - 1$, which contradicts the assumed convergence. $\square$

Corollary. *All Toeplitz flows constructed in Section 3 are coalescent.*

Proof. The statement follows for Toeplitz flows with entropy zero, since they all have separated holes. For the modified family with nonzero entropy apply a coalescent Toeplitz flow on G_q and then observe the iterate of each endomorphism which preserves the copies of G_q and G_p in G_r. $\square$

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