On Strong Uniform Distribution, III

By

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Abstract. We construct infinite dimensional chains that are $L^1$ good for almost sure convergence, which settles a question raised in this journal [7] and earlier in [6] by R. Nair. In [7] it was stated that the construction proposed in [4] was invalid. We complete the construction proposed in [4], where it is true that a piece of proof was forgotten. The technic remains the same and the completion of the proof rather natural.

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1. Introduction

A chain $C$ is a multiplicative sub-semigroup of the one of positive integers $\mathbb{N}$. We say a sequence $p = (p_k)$ of primes generates the chain $C$ if $C = \left\{ \prod_{j=1}^{J} p_k^{\alpha_j} : \alpha_k \geq 0, J \geq 1 \right\}$. A chain is of finite dimension (abbreviated “an FD chain”) if the sequence of primes generating it is finite; else, it is infinite dimensional (abbreviated “ID chain”).

Throughout, $([\mathbb{T}, \lambda])$ denotes the reals mod 1 with Lebesgue measure. We shall say that a chain $C$ is good if, once one orders $C = \{a_1 < a_2 < \cdots \}$, it holds that for any $f \in L^1([\mathbb{T}, \lambda])$,

$$\frac{1}{k} \sum_{j=1}^{k} f(a_j x \mod 1) \to \int_{0}^{1} f d\lambda \quad \text{for} \lambda - \text{a.e.} \ x \in \mathbb{T}. \quad (1)$$

Else we say $C$ is a bad chain.

Nair [6, 7] asks twice for the existence of a good ID chain. He proves in [6] that an FD chain is always good, using the multidimensional ergodic theorem ([1], [3]) for $\mathbb{N}^d$-actions. It is also known ([2], [5]) that taking all the primes generates a bad ID chain, with counter-examples to almost sure convergence in (1) for some $f \in L^\infty([\mathbb{T}, \lambda])$.

In [7] one can find on page 342, a few lines after formula [7, (1.3)], the following (we have adapted the reference numbering):

“In [6] the author raised the question whether the condition in his theorem that the set $p_1, \ldots, p_d$ be finite is necessary. This question remains open despite the invalid construction of a putative such infinite set in [4].”
A PhD student of the author, Vincent Chaumoître, has, following [7], made a precise rereading of [4], and pointed out to the author the precise spot where [4] was uncomplete. In fact, the displayed formula between [4, (5)] and [4, (P6)] is only correct in the FD case, and hence the reduction of [4, (P2)] to [4, (P6)] is not valid in the ID case. The place where this omission occurs in [4] corresponds to the part of the paper devoted to show how the result from [6] could be recovered using Tempelman’s ergodic theorem, giving a simple proof that an FD chain is good [4, Corollary 1] (cf. [6] for a different approach).

The present note completes the gap, using exactly the same ideas as in [4] to produce a complete proof of the following:

**Theorem 1.** There exist good ID chains.

We will present the completed argumentation omitting the ergodic theoretic preliminaries for which we refer to [4]. Let us mention by the way that the construction of bad ID chains in [4] is perfectly valid.

2. Good ID Chains!

2.1. Semigroup actions and Tempelman’s conditions. We shall make essential use of the following abelian semi-group endowed with its counting measure (for a subset $T$, $\#T$ denotes its cardinality):

$$l_0(\mathbb{N}) := \{(\alpha_i)_{i \geq 1} : \alpha_i \in \mathbb{N}, \exists j, \ i > j \Rightarrow \alpha_i = 0\}.$$  

Given an integer $q$, we identify $\mathbb{N}^q$ with a sub-semigroup of $\mathbb{N}^{q+1}$ and the later with one of $l_0(\mathbb{N})$ via the following embeddings:

$$\begin{align*}
\mathbb{N}^q &\leftrightarrow \mathbb{N}^{q+1} & \leftrightarrow & \quad l_0(\mathbb{N}) \\
(\alpha_1, \ldots, \alpha_q) &\mapsto (\alpha_1, \ldots, \alpha_q, 0) & \mapsto & (\alpha_1, \ldots, \alpha_q, 0, 0, \ldots).
\end{align*}$$

For an integer $p$, we define $T_p : X \rightarrow X$ by $T_p x = px \mod 1$. It is standard that the system $(\mathbb{T}, \lambda, T_p)$ is metrically conjugated to a one sided Bernoulli shift which is ergodic [3]. It is standard also that $T_p \circ T_q = T_q \circ T_p = T_{pq}$, whence given a sequence of integers $(p_k)_{k \geq 1}$, we define an action $\Gamma$ of $l_0(\mathbb{N})$ on $(\mathbb{T}, \lambda, T_p)$ by

$$\Gamma((\alpha_k)) := \bigcirc_{k \geq 1} T_{p_k}^{\alpha_k},$$

where $T_{p_k}^0$ is the identity map. Given any sequence $(T(n))$ of subsets of $l_0(\mathbb{N})$, we will consider the following multiple condition $(P)$:

$$\begin{align*}
(P1) : & \quad 0 < \#T(n) < \infty, \\
(P2) : & \quad \forall \gamma \in l_0(\mathbb{N}), \lim_n \#((T(n) + \gamma)\Delta T(n))/\#T(n) = 0, \\
(P3) : & \quad T(n) \subset T(n+1), n \geq 1, \\
(P4) : & \quad \exists K_1 < \infty, \forall N, \lim_n \#(T(N) + T(n))/\#T(n) \leq K_1, \\
(P5) : & \quad \exists K_2 < \infty, \forall n, \#(T(n) - T(n))/\#T(n) \leq K_2,
\end{align*}$$

where $T(n) - T(n) := \{\alpha \in l_0(\mathbb{N}) : \exists \gamma \in T(n), \ \alpha + \gamma \in T(n)\}$. 

Indeed, if \((T(n))\) satisfies \((P)\), by Tempelman’s Ergodic Theorem [3, p. 224], for any \(f \in L^1(\mu)\), the averages

\[
\frac{1}{\#T(n)} \sum_{\alpha \in T(n)} f \circ \Gamma(\alpha)(x)
\]

converge \(\mu\text{-a.e.}\). Moreover, the limit in (2) is, following the argument in [3, p. 206], or [8, Theorem 6.3.1], a \(\Gamma\)-invariant function, whence \(T_p\)-invariant for some \(p \geq 2\), whence, by ergodicity of \(T_p\), it is constant and must coincide with the expectation of \(f\), as is standard.

When (2) holds we say that \((T(n))\) is \(L^1\) good universal (for \(l_0(\mathbb{N})\) actions). We shall see in the next section that for some choice of \((T(n))\), therefore the condition \((P)\) will be used to produce a good ID chain.

The same remarks can be stated for \(\mathbb{N}^q\)-actions.

2.2. Condition \((P)\) for a pairwise coprime generated chain and the FD case.

Let \(p_1 < p_2 < \cdots\) be pairwise coprime integers generating the chain \(\mathcal{C} = \{a_1 < a_2 < \cdots\}\). For given \(q \geq 1\) and \(n \in [1, \infty]\), we let

\[
\begin{align*}
T_q(n) & := \{ (\alpha_1, \ldots, \alpha_q) \in \mathbb{N}^q : \sum_{i=1}^q \alpha_i \log p_i \leq \log n \}, \\
T(n) & := \{ \alpha = (\alpha_i) \in l_0(\mathbb{N}) : \sum_{i \geq 1} \alpha_i \log p_i \leq \log n \}. 
\end{align*}
\]

We notice that sequences (1) and (2) coincide for this choice of \((T(n)), (T_q(n))\) in the FD case). For given \(q \geq 1\), both \((T_q(n))\) and \((T(n))\) satisfy \((P1), (P3),\) and \((P5)\) with \(K_2 = 1\), because \(T(n) - T(n) \subset T(n)\).

Moreover, since \(T(n) \subset T(N) + T(n)\) (resp. \(T_q(n) \subset T_q(N) + T_q(n)\)), we have

\[
\begin{align*}
\#(T(N) + T(n)) & \leq \#T(n) + \sum_{\gamma \in T(N)} \#((\gamma + T(n)) \setminus T(n)) \\
(\text{resp. } \#(T_q(N) + T_q(n)) & \leq \#T_q(n) + \sum_{\gamma \in T_q(N)} \#((\gamma + T_q(n)) \setminus T_q(n)))
\end{align*}
\]

so we see that \((P2)\) implies \((P4)\) with \(K_1 = 1\). Hence we deduce

**Lemma 1.** The sequence \((T(n))\) (resp. \((T_q(n))\)) defined by (3) is \(L^1\) good universal for \(l_0(\mathbb{N})\) (resp. \(\mathbb{N}^q\)) actions whenever it satisfies \((P2)\).

Given \(\gamma = (\gamma_i) \in l_0(\mathbb{N})\) (resp. \(\gamma \in \mathbb{N}^q\)), we have

\[
\begin{align*}
\#((T(n) + \gamma) \Delta T(n)) & = \#(T(n) \setminus (T(n) + \gamma)) + \#((T(n) + \gamma) \setminus T(n)) \\
(\text{resp. } \#(T_q(n) + \gamma) \Delta T_q(n)) & = \#(T_q(n) \setminus (T_q(n) + \gamma)) + \#((T_q(n) + \gamma) \setminus T_q(n)))
\end{align*}
\]

An elementary computation [5] shows that

\[
\#T_q(n) \sim \frac{(\log n)^q}{q! \prod_{i=1}^q \log p_i},
\]

where \(\sim\) means that the ratio of its left and right hand sides goes to 1 as \(n\) goes to \(\infty\).

For any \(\gamma \in \mathbb{N}^q, T_q(n) \setminus (T_q(n) + \gamma) = \{ \alpha \in T_q(n) : \exists i : \alpha_i < \gamma_i \}, \) so if we set

\[
\begin{align*}
B_q(n, i) & = \{ \alpha \in T_q(n) : \alpha_i < \gamma_i \} \quad \text{and} \\
T_{q[i]}(n) & = \{ (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_q) \in \mathbb{N}^{q-1} : \sum_{j \neq i} \alpha_j \log p_j \leq \log n \},
\end{align*}
\]
then one observes that
\[
\#(T_q(n) \setminus (T_q(n) + \gamma)) \leq \sum_{i=1}^{q} \#B_q(n, i) \leq \sum_{i=1}^{q} \gamma_i \#T_q^{(i)}(n),
\]
whence by (5) we get at once that \(\lim_n \#(T_q(n) \setminus (T_q(n) + \gamma)) / \#T_q(n) = 0\). Secondly, we have that
\[
(T_q(n) + \gamma) \setminus T_q(n) = \{ \alpha + \gamma : \sum_{i} \alpha_i \log p_i \leq \log n \text{ and } \sum_{i} (\alpha_i + \gamma_i) \log p_i > \log n \} \subseteq \{ \beta \in \mathbb{N}^q : \log n < \sum_{i} \beta_i \log p_i < \log (n \times n(\gamma)) \},
\]
where \(n(\gamma) = \prod_i p_i^{\gamma_i}\). So since \(T_q(n) \subseteq T_q(n \times n(\gamma))\), we deduce that
\[
\frac{\#((T_q(n) + \gamma) \setminus T_q(n))}{\#T_q(n)} \leq \frac{\#T_q(n \times n(\gamma)) - \#T_q(n)}{\#T_q(n)} \to 0 \text{ by (5)},
\]
hence with (4), (P2) is satisfied for the \(\mathbb{N}^q\) case, and as a consequence of our study of the FD case we obtain

Corollary 1. ([6, Theorem 1]) Any FD chain satisfies (P2), whence is good.

2.3. The inductive step for constructing a good ID chain. We set for \(\gamma \in l_0(\mathbb{N})\) (resp. \(\gamma \in \mathbb{N}^q\))
\[
\partial_\gamma(T(n)) = (T(n) + \gamma) \Delta T(n) \text{ (resp. } \partial_\gamma(T_q(n)) = (T_q(n) + \gamma) \Delta T_q(n)).
\]

We know by Lemma 1 that (P2) is enough for an ID coprime generated chain to be good. And we also know by Corollary 1 that (P2) holds in the FD case. The idea to reach the ID case is to show that given \(p_1 < \cdots < p_q\), it is possible to choose \(p_{q+1} > p_q\) such that “small” increase occurs in the quotients (P2) uniformly in \(\gamma\) belonging to some finite subset \(\langle q \rangle\) of \(l_0(\mathbb{N})\), where the increasing union over \(q\) of these subsets cover \(l_0(\mathbb{N})\). This is done in the present section and summarized in Lemma 2 below.

If \(q(n) := \max\{ q : p_q \leq n \}\), then \(T(n) = T_{q(n)}(n)\). Our argumentation shall strongly rely on this equality, on a careful use of (5) and the second estimate (cf. [5] where it is proved for the first \(q\) primes but carries out also in the case we need here)
\[
\#((T_q(n) + \bar{q}) \setminus T_q(n)) \sim_{x \to \infty} \frac{\log (p_1^{q} \cdots p_q^{q})}{(q - 1)! \prod_{i=1}^{q} \log p_i} (\log x)^{q - 1}, \tag{6}
\]
where \(\bar{q} = (q, q, \ldots, q)\).

We assume \(q > 1\) and that \(p_1 < \cdots < p_q\) are pairwise coprime. We define
\[
\langle q \rangle := \{ \gamma = (\gamma_i) \in \mathbb{N}^q : \gamma_i \leq q, \ 1 \leq i \leq q \}.
\]
Given arbitrary \(\varepsilon_q > 0\), by (P2) for the FD case (Corollary 1), there exists an \(N(\varepsilon_q)\) such that
\[
x \geq N(\varepsilon_q) \Rightarrow \forall \gamma \in \langle q \rangle, \#\partial_\gamma(T_q(x)) / \#T_q(x) < \frac{\varepsilon_q}{2}. \tag{7}
\]

We now let \(p_{q+1} > p_q\) denote an integer coprime to the previous numbers \(p_k\), to be specified later on. We assume that \(p_{q+1} \geq N(\varepsilon_q)\) (\(N(\varepsilon_q)\) comes in (7)). Then if
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\[ k \geq 1 \) and \( p_{q+1}^k \leq n < p_{q+1}^{k+1} \), we have

\[ T_{q+1}(n) = \sum_{i=0}^{k} (\mathbb{N}^q \times \{i\}) \cap T_{q+1}(n) \] (a disjoint union).

Let us put \( T_{q+1}(n, i) := (\mathbb{N}^q \times \{i\}) \cap T_{q+1}(n), 0 \leq i \leq k = \left\lfloor \frac{\log n}{\log p_{q+1}} \right\rfloor \). Then we observe that

\[ (\alpha_1, \ldots, \alpha_q, i) \in T_{q+1}(n, i) \Leftrightarrow (\alpha_1, \ldots, \alpha_q) \in T_q \left( \frac{n}{p_{q+1}^i} \right), \]

and moreover if \( \gamma \in \langle q \rangle \subseteq \mathbb{N}^q \), then \( \gamma_{q+1} = 0 \), and therefore \( \partial_\gamma(T_{q+1}(n)) = \sum_{i=0}^{k} \partial_\gamma(T_{q+1}(n, i)) \) (disjoint union) where \( \partial_\gamma(T_{q+1}(n, i)) = (T_{q+1}(n, i) + \gamma) \Delta T_{q+1}(n, i) \). Then for such \( \gamma \),

\[ (\alpha_1, \ldots, \alpha_q, i) + \gamma \in \partial_\gamma(T_{q+1}(n)) \Leftrightarrow (\alpha_1, \ldots, \alpha_q) + \gamma \in \partial_\gamma \left( T_q \left( \frac{n}{p_{q+1}^i} \right) \right). \]

We now define \( (\gamma \leq \gamma') \Leftrightarrow (\forall i, \; \gamma_i \leq \gamma'_i) \). An easy observation is

\[ \gamma \leq \gamma' \Rightarrow \#\partial_\gamma(T_q(n)) \leq \#\partial_\gamma(T_q(n)). \]

Hence with the above we get

\[ \gamma \in \langle q \rangle \Rightarrow \\begin{cases} \#T_{q+1}(n) = \sum_{i=0}^{k} \#T_q \left( \frac{n}{p_{q+1}^i} \right), \\ \#\partial_\gamma(T_{q+1}(n)) \leq \#\partial_\gamma(T_q(n)) = \sum_{i=0}^{k} \#\partial_\gamma \left( T_q \left( \frac{n}{p_{q+1}^i} \right) \right). \end{cases} \]

Therefore as soon as \( n, p_{q+1} \geq N(\varepsilon_q) \), if \( k = \left\lfloor \frac{\log n}{\log p_{q+1}} \right\rfloor \), we have, using (7):

\[ k = 0 \) (i.e. \( N(\varepsilon_q) \leq n < p_{q+1} \)) \Rightarrow T_{q+1}(n) = T_q(n) \]

\[ \Rightarrow \forall \gamma \in \langle q \rangle, \; \#\partial_\gamma(T_{q+1}(n))/\#T_{q+1}(n) < \frac{\varepsilon_q}{2} , \]

and

\[ k \neq 0 \Rightarrow \forall \gamma \in \langle q \rangle, \]

\[ \#\partial_\gamma(T_{q+1}(n))/\#T_{q+1}(n) \leq \#\partial_\gamma(T_q(n))/\#T_{q+1}(n) \]

\[ \leq \frac{\sum_{i=0}^{k-1} \#\partial_\gamma \left( T_q \left( \frac{n}{p_{q+1}^i} \right) \right)}{\sum_{i=0}^{k-1} \#T_q \left( \frac{n}{p_{q+1}^i} \right)} + \frac{\#\partial_\gamma \left( T_q \left( \frac{n}{p_{q+1}^k} \right) \right)}{\#T_q \left( \frac{n}{p_{q+1}^k} \right)} \]

\[ < \frac{\varepsilon_q}{2} + A \left( p_{q+1}, \frac{n}{p_{q+1}^k} \right), \]

where \( A(p_{q+1}, x) = \#\partial_\gamma(T_q(x))/T_q(p_{q+1}x) \) \((x \geq 1)\).

Next we can write

\[ A(p_{q+1}, x) = \frac{\#(T_q(x) + \bar{q}) \setminus T_q(x)}{\#T_q(p_{q+1}x)} + \frac{\#(T_q(x) \setminus (T_q(x) + \bar{q}))}{\#T_q(p_{q+1}x)}. \]

We firstly can estimate as in Section 2.2 that

\[ \#(T_q(x) \setminus (T_q(x) + \bar{q})) \leq q \sum_{i=1}^{q} \#T_q(i)(x) \leq q \sum_{i=1}^{q} \#T_q(i)(p_{q+1}x), \]
which makes sure that, using (5), \( \frac{\#((T_q(x) + \bar{q}) \setminus T_q(x))}{\#T_q(p_{q+1}x)} \rightarrow 0 \) as \( p_{q+1} \rightarrow +\infty \), uniformly in \( x \geq 1 \).

Secondly, by (5, 6), there exist two positive constants \( C_1 \) and \( C_2 \), depending only on \( q \), such that uniformly in \( x \geq 1 \) and \( p_{q+1} \),

\[
\begin{align*}
\#((T_q(x) + \bar{q}) \setminus T_q(x)) &\leq C_1 \log(x)^{q-1}, \text{ by (6)} \\
\#T_q(p_{q+1}x) &\geq C_2 \log(p_{q+1}x)^{q}, \text{ by (5)}
\end{align*}
\]

whence there exists some positive constant \( C \) depending only on \( p_1, \ldots, p_q \) such that uniformly in \( x \geq 1 \), for any \( p_{q+1} \),

\[
\frac{\#((T_q(x) + \bar{q}) \setminus T_q(x))}{\#T_q(p_{q+1}x)} \leq \frac{C}{\log p_{q+1}}.
\]

Finally we may select \( n(q) \geq N(\varepsilon_q) \) so large that uniformly in \( x \geq 1 \),

\[
A(p_{q+1}, x) < \frac{1}{2} \varepsilon_q.
\]

For such choice of \( p_{q+1} \), we get that as soon as \( n \geq N(\varepsilon_q) \), for any \( \gamma \in \langle q \rangle \),

\[
\#\partial_\gamma(T_{q+1}(n)) / \#T_{q+1}(n) < \varepsilon_q,
\]

we have proved:

**Lemma 2.** Given \( q > 1 \), arbitrary coprime \( p_1 < \cdots < p_q \), arbitrary \( \varepsilon_q > 0 \), there exists an integer \( N(\varepsilon_q) \) and a \( p_{q+1} \geq N(\varepsilon_q) \) which is coprime to the \( p_i \)'s \((1 \leq i \leq q)\), such that for any \( \gamma \in \langle q \rangle \), if \( n \geq N(\varepsilon_q) \), (8) holds.

### 2.4. The inductive construction of good ID chains.

We fix a sequence \( \langle \varepsilon_q \rangle \geq 1 \)
of positive real numbers tending to 0. Next we select arbitrary \( p_1 > 0 \). Then a repeated inductive use of Lemma 2 produces a sequence \( p_1 < p_2 < p_3 < \cdots < p_{q+1} < \cdots \) of pairwise coprime integers, and another sequence \( \langle N(\varepsilon_i) \rangle \leq N(\varepsilon_2) \leq \cdots \leq N(\varepsilon_q) \leq \cdots \) of integers (we can choose them increasing), along with the corresponding properties in (8).

We then define, for each \( n \), the set \( T(n) \) as in (3). As before, \( T(n) = T_{q(n)}(n) \), where \( p_{q(n)} \leq n < p_{q(n)+1} \); then if \( n > p_2 \) (that is \( q(n) \geq 2 \)), and \( q < q(n) - 1 \),

\[
\gamma \in \langle q \rangle \subset \langle q(n) - 1 \rangle \Rightarrow \#\partial_\gamma(T_{q(n)}(n)) / \#T_{q(n)}(n) < \varepsilon_{q(n) - 1},
\]

because \( p_{q(n)} \) exceeds \( N(\varepsilon_{q(n) - 1}) \).

Now we fix \( \gamma \in l_0(\mathbb{N}) \) and select \( q \geq 2 \) such that \( \gamma \in \langle q \rangle \). Then if \( n_0 \) satisfies \( q(n_0) - 1 \geq q \), we obtain that for any \( n \geq n_0 \),

\[
\#\partial_\gamma(T(n)) / \#T(n) = \#\partial_\gamma(T_{q(n)}(n)) / \#T_{q(n)}(n) < \varepsilon_{q(n) - 1},
\]

by our inductive construction using Lemma 2. Since \( \varepsilon_{q} \rightarrow 0 \) and \( q(n) \rightarrow \infty \), this proves \( (P2) \). By Lemma 1, we have proved Theorem 1.

### References


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